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ORIGINAL PAPER

Generalized extragradient iterative methods for solving split feasibility and fixed point problems in Hilbert spaces

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Abstract
In the paper, we analyze the extragradient method with regularization for finding a common element of the solution set of the split feasibility and fixed point problems of pseudo-contractive mappings. Moreover we propose the extragradient with regularization iterative method due to a generalized Ishikawa-type and Mann-type iterative methods. The weak convergence theorems of the sequences generated by the proposed iterative methods are obtained under the certain assumptions on pseudo-contractive mappings in real Hilbert spaces. Finally, we give the numerical example to demonstrate the effectiveness of our theoretical results and compare its behavior with the iterative methods of Ceng et al. (Fixed Point Theory Appl 192, 2015).

Keywords Extragradient method · Regularization · Pseudocontractive mapping · Split feasibility problem · Fixed point problem

Mathematics Subject Classification 47J06 · 47H10 · 65K10

1 Introduction

Throughout the paper, we assume H_1 and H_2 are real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Denote the strong convergence by $\to$ and weak convergence by $\rightharpoonup$. A mapping $S : C \to C$ is a nonexpansive mapping if

$$\|Sx - Sy\| \leq \|x - y\| \quad \text{for all} \quad x, y \in C.$$

On the other hand, in a real Hilbert space, a mapping $T : C \to C$ is called pseudo-contractive if,

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 \quad \text{for all} \quad x, y \in C.$$

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It is well-known that T is pseudo-contractive if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \quad \text{for all} \quad x, y \in C.$$

The fixed point problem for the mapping T is the following:

$$\text{find} \quad x \in C \quad \text{such that} \quad Tx = x.$$

Denote by $F(T) = \{x \in C : Tx = x\}$ the set of solutions of the fixed point problem.

In 1953, Mann [31] introduced the Mann iterative method as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \text{for all } n \in \mathbb{N}, \tag{1}$$

where $\{\alpha_n\} \subset [0, 1]$.

And then in 1974, Ishikawa [28] introduced the Ishikawa iterative method as follows:

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n, \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$.

Next, Noor [33] introduced three-step iterative method as follows:

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \\ z_n = (1 - \beta_n)x_n + \beta_n T y_n, \\ x_{n+1} = (1 - \gamma_n)x_n + \gamma_n T z_n, \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1]$. Clearly, Mann and Ishikawa iterative methods are special cases of Noor iteration.

The above iterative methods have been extensively studied by many authors for approximating fixed points of nonlinear mappings and solutions of nonlinear operator equations.

On the other hand, the split feasibility problems (SFP) have the following property:

$$\text{find} \quad x \in C \quad \text{such that} \quad Ax \in Q.$$

where $A : H_1 \to H_2$ is a bounded linear operator. Denote $\Gamma_0 = \{x \in C : Ax \in Q\}$ the set of solutions of the split feasibility problems (SFP) and $\Gamma = \{x \in F(T) \cap C : Ax \in F(S) \cap Q\}$ the set of solutions of the split feasibility and fixed point problems where $T : C \to C$ and $S : Q \to Q$.

Censor and Elfving [21] introduced the split feasibility problems (SFP) in finite-dimension Hilbert spaces for modeling inverse problems which arise in phase retrievals and medical image reconstruction [1]. The split feasibility problems (SFP) can also be applied to intensity-modulated radiation therapy (IMRT) [22–24] and have been used in signal processing and image reconstruction, see [1,2,22,34,37,39,46].

The original iterative method for solving the split feasibility problems (SFP) is given in [21] under assuming the existence of the inverse of A. We know that the finding of the inverse of A is difficult so this iterative method has not become popular. A more popular iterative method for solving the split feasibility problems (SFP) is the CQ iterative method which is introduced by Byrne [21] because it is found to be a gradient-projection method (GPM) in the convex minimization and a special case of the proximal forward-backward splitting method [27].

Many researchers have studied the CQ iterative method and its variant form, refer to [10,12,17,36,38,40,41,45]. In 2010, Xu [38] applied a Mann-type iterative method to the split feasibility problems (SFP) and proposed an average CQ iterative method which was proven to be weakly convergent to a solution of the split feasibility problems (SFP).

In 1976, Korpelevich [30] introduced the extra-gradient iterative method for solving a saddle point problems such that many researchers have used and applied this iterative method for solving various problems: see e.g. [3–6,11,13,17–20]

For solving the split feasibility and fixed point problems, in 2012, Ceng et al. [12] proposed an iterative method by combining the extragradient iterative method with the idea of Nadezhkina and Takahashi [32] and proved that the sequences generated by their iterative method converge weakly to an element of the solutions of the split feasibility and fixed point problems.

In 2014, Yao et al. [42] studied the split feasibility and fixed point problems. They [42] constructed an iterative method as the following:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = P_C(\alpha_n u + (1-\alpha_n)(x_n - \delta A^*(I - SP_Q)Ax_n)), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T((1-\gamma_n)y_n + \gamma_n T y_n), \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three real number sequences in $(0, 1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$. They [42] proved that the sequences generated by their iterative method converge strongly to solutions of the split feasibility and fixed point problems.

Very recently, Ceng et al. [25] had the motivation and inspiration from the work of Ceng et al. [12] and Yao et al. [42]. They proposed an Ishikawa-type extragradient iterative method for pseudo-contractive mappings with Lipschitz assumption on T. For given $x_0 \in C$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ax_n), \\ z_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ay_n), \\ w_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = (1-\beta_n)z_n + \beta_n T w_n, \quad \text{for all } n \in \mathbb{N}. \end{cases} \tag{5}$$

Moreover, they proposed a Mann-type extragradient iterative method for pseudo-contractive mappings without Lipschitz assumption on T as the following:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ax_n), \\ z_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ay_n), \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n T z_n, \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{6}$$

where $S : Q \to Q$ is a nonexpansive mapping, $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^* and they [25] proved that their sequences generated by their iterative methods converge weakly to solutions of the split feasibility and fixed point problems.

Throughout this research, we assume that the solution set of the split feasibility problems are nonempty. Let $f : H_1 \to \mathbb{R}$ be a continuous differentiable function, the minimization problem:

$$\min_{x \in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 \tag{7}$$

is ill-posed. Therefore, Xu [38] considered the following Tikhonov regularized problem:

$$\min_{x \in C} f^\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{8}$$

where $\alpha > 0$ is the regularization parameter.

We observe that the gradient

$$\nabla f^\alpha(x) = \nabla f(x) + \alpha I = A^*(I - P_Q)A + \alpha I$$

is $(\alpha + \|A\|^2)$-Lipschitz continuous and α-strongly monotone.

It is worth to emphasize that the traditional Tikhonov regularization is usually used to solve the ill-posed optimization problems. The advantage of a regularization method is its possible strong convergence to the minimum-norm solution of the optimization problems: see e.g. [7,15–17].

In 2012, Ceng et al. [11] proposed iterative method by combining the regularization method and extragradient method due to Nadezhkina and Takahashi [32] and proved that the sequence generated by their iterative method converges weakly to an element of the solution of the split feasibility and fixed point problems.

Motivated and inspired by the research mentioned above, we introduce the iterative methods by using a combination of an extragradient method with regularization due to a generalized Ishikawa iterative method for solving the split feasibility and the fixed point problems of pseudo-contractive mappings with Lipschitz assumption on C and nonexpansive mappings on Q. On the other hand, we avoid Lipschitzian condition by a proposed iterative method which combine an extragradient method with regularization due to a generalized Mann iterative method for solving the split feasibility and fixed point problems. We establish weak convergence theorems for sequences generated by the proposed iterative processes. Finally, we give numerical results and compare its behavior with an Ishikawa-type extragradient iterative method and a Mann-type extragradient iterative method of Ceng et al. [25].

2 Preliminaries

Let C be a closed convex subset of a Hilbert space H. The mapping $P_c : H \to C$ is called the metric projection if $P_C x$ is the unique point in C with the property:

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\} \quad \text{for all } x \in H.$$

Some properties of the metric projection which we use in our main results appeared in the following proposition.

Proposition 1 *Let $x \in H$ and $z \in C$. Then*

(1) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0$ *for all* $y \in C$;
(2) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$ *for all* $y \in C$;
(3) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$ *for all* $y \in C$.

We also need other properties of nonlinear operators as the following:
a nonlinear operator $T : H \to H$ is said to be

(1) L-Lipschitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| < L\|x - y\|, \quad \text{for all } x, y \in H,$$

if $L = 1$, then T is called nonexpansive;
(2) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \text{for all } x, y \in H,$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive;

(3) monotone if

$$\langle x-y, Tx-Ty\rangle \geq 0, \quad \text{for all } x, y \in H;$$

(4) β-strongly monotone with $\beta > 0$, if

$$\langle x-y, Tx-Ty\rangle \geq \beta\|x-y\|^2, \quad \text{for all } x, y \in H;$$

(5) ν-inverse strongly monotone (ν-ism), with $\nu > 0$, if

$$\langle x-y, Tx-Ty\rangle \geq \nu\|Tx-Ty\|^2 \quad \text{for all } x, y \in H.$$

It is well-known that the metric projection $P_C : H \to C$ is firmly nonexpansive, that is,

$$\begin{aligned}&\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2\\ &\quad\Leftrightarrow \|P_Cx-P_Cy\|^2 \leq \|x-y\|^2 - \|(I-P_C)x-(I-P_C)y\|^2, \quad \forall x, y \in H. \end{aligned} \tag{9}$$

For all $x, y, z \in H$, we have

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{10}$$

$$\|\alpha x+(1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \quad \text{where } \alpha \in [0,1], \tag{11}$$

and

$$\begin{aligned}\|\alpha x+\beta y+\gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2\\ &+\gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2,\end{aligned} \tag{12}$$

where $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

Proposition 2 *Let $T : H \to H$ be a given mapping. Then*

(1) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism;*
(2) *if T is ν-ism, then γT is $\frac{\nu}{\gamma}$-ism, for $\gamma > 0$;*
(3) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Proposition 3 [25] *Let T be a pseudo-contractive mapping with the nonempty fixed point set $F(T)$, then the following conclusion holds:*

$$\langle Ty-y, Ty-x^*\rangle \leq \|Ty-y\|^2, \quad \textit{for all } y \in C, \quad x^* \in F(T).$$

In general, many researches have assumed pseudo-contractive mappings with L-Lipschitzian with $L > 1$. Ceng et al. [25] overcome the L-Lipschitzian property by assuming the condition of the pseudo-contractive mapping on T:

$$\langle Ty-y, Ty-x^*\rangle \leq 0, \quad \text{for all } y \in C, \quad x^* \in F(T). \tag{13}$$

In our main result, we use the demiclosedness principle for pseudo-contractive mappings.

Definition 1 Let $T : H \to H$ be a mapping. A mapping $I - T$ is said to be demiclosed at zero if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, and $x_n - Tx_n \to 0$, we have $x = Tx$.

Lemma 1 [47] *Let H be a real Hilbert space, C be a closed convex subset of H. Let $T : C \to C$ be a continuous pseudo-contractive mapping. Then*

(1) *$F(T)$ is a closed convex subset of C;*
(2) *$I - T$ is demiclosed at zero.*

Moreover, we use a weak-cluster point of the sequence $\{x_n\}$, denote by

$$\omega_W(x_n) = \left\{ x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\} \right\}.$$

Lemma 2 [29] *Let H be a real Hilbert space and $\{x_n\}$ be a bounded sequence in H such that there exists a nonempty closed convex set C of H satisfying:*

(1) *for every $w \in C$, $\lim_{n\to\infty} \|x_n - w\|$ exists;*
(2) *each weak-cluster point of the sequence $\{x_n\}$ is in C.*

Then $\{x_n\}$ converges weakly to a point in C.

Lemma 3 [25] *Let Q be a nonempty closed convex subset of a Hilbert space H and $S : Q \to Q$ be a nonexpansive mapping. Set $\nabla f^S = A^*(I - SP_Q)A$, then*

$$\langle x - y, \nabla f^S(x) - \nabla f^S(y)\rangle \geq \frac{1}{2\|A\|^2} \|\nabla f^S(x) - \nabla f^S(y)\|^2. \tag{14}$$

We can use the fixed point algorithms to solve the split feasibility problems on the basis of the following observation.

Let $\lambda > 0$ and assume that $x^* \in \Gamma$. Then $Ax^* \in Q$ which implies that $(I - P_Q)Ax^* = 0$, and thus, $\lambda(I - P_Q)Ax^* = 0$. Hence, we have the fixed point equation $x^* = (I - \lambda A^*(I - P_Q)A)x^*$. Requiring that $x^* \in C$, we consider the fixed point equation

$$x^* = P_C(I - \lambda A^*(I - P_Q)A)x^* = P_C(I - \lambda\nabla f)x^*. \tag{15}$$

It is proven in [38] that the solutions of the fixed point equation (15) are exactly the solutions of the split feasibility problems; namely, for given $x^* \in C$, x^* solves the split feasibility problems if and only if x^* solves the fixed point equation (15).

3 Main results

We propose the generalized Ishikawa-type extragradient with regularization iterative method for pseudo-contractive mappings with Lipschitz assumption and the generalized Mann-type extragradient with regularization iterative method for pseudo-contractive mappings without Lipschitz assumption for solving the split feasibility and fixed point problems.

3.1 The generalized Ishikawa-type extragradient with regularization iterative method for pseudo-contractive mappings with Lipschitz assumption

Theorem 1 *Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a* *sequence defined by*

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\right), \\ z_n = P_C\left(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)y_n\right), \\ w_n = (1-\sigma_n)z_n + \sigma_n T z_n, \\ s_n = (1-\beta_n)z_n + \beta_n T w_n, \\ x_{n+1} = (1-\gamma_n)z_n + \gamma_n T s_n, \end{cases} \tag{16}$$

where $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ *and* $0 < a < \gamma_n < b < \beta_n < c < \sigma_n < d < \frac{1}{\sqrt{L^2+1}+1+L^2}$. *Then the sequence* $\{x_n\}$ *generated by algorithm* (16) *converges weakly to an element of* Γ.

Proof Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in F(T) \cap C$ and $Ax^* \in F(S) \cap Q$. Set $v_n = P_Q A x_n$, $u_n = x_n - \lambda_n(A^*(I - SP_Q)Ax_n + \alpha_n I)x_n$, $\nabla f^{S\alpha_n} = A^*(I - SP_Q)A + \alpha_n I$ and $\nabla f^S = A^*(I - SP_Q)A$, for all $n \ge 0$. Since P_C is nonexpansive, we have

$$\begin{aligned} \|y_n - x^*\|^2 &= \|P_C u_n - x^*\|^2 \le \|u_n - x^*\|^2 \\ &= \|x_n - \lambda_n\left(A^*(I - SP_Q)A + \alpha_n I\right)x_n - x^*\|^2 \\ &= \|x_n - x^*\|^2 + 2\lambda_n\langle x_n - x^*, A^*(SP_Q - I)Ax_n\rangle \\ &\quad + \lambda_n^2\|A^*(SP_Q - I)Ax_n\|^2 - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle. \end{aligned} \tag{17}$$

From A is a linear operator with its adjoint A^*, we obtain that

$$\begin{aligned} \langle x_n - x^*, A^*(Sv_n - Ax_n)\rangle &= \langle Ax_n - Ax^*, Sv_n - Ax_n\rangle \\ &= \langle Ax_n - Ax^* + Sv_n - Ax_n - Sv_n + Ax_n, Sv_n - Ax_n\rangle \\ &= \langle Sv_n - Ax^*, Sv_n - Ax_n\rangle - \|Sv_n - Ax_n\|^2. \end{aligned} \tag{18}$$

In combination with (11), we get that

$$\langle Sv_n - Ax^*, Sv_n - Ax_n\rangle = \frac{1}{2}(\|Sv_n - Ax^*\|^2 + \|Sv_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2). \tag{19}$$

Since S is a nonexpansive mapping and (9), we have

$$\begin{aligned} \|Sv_n - Ax^*\|^2 &= \|SP_Q Ax_n - SP_Q Ax^*\|^2 \\ &\le \|P_Q Ax_n - P_Q Ax^*\|^2 \\ &\le \|Ax_n - Ax^*\|^2 - \|v_n - Ax_n\|^2. \end{aligned} \tag{20}$$

In view of (18), (19) and (20), it follows that

$$\begin{aligned} &\langle x_n - x^*, A^*(Sv_n - Ax_n)\rangle \\ &= \frac{1}{2}(\|Sv_n - Ax^*\|^2 + \|Sv_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2) - \|Sv_n - Ax_n\|^2 \\ &\le \frac{1}{2}(\|Ax_n - Ax^*\|^2 - \|v_n - Ax^*\|^2 + \|Sv_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2) \\ &\quad - \|Sv_n - Ax_n\|^2 \\ &= -\frac{1}{2}\|v_n - Ax^*\|^2 - \frac{1}{2}\|Sv_n - Ax_n\|^2. \end{aligned} \tag{21}$$

Substituting (21) into (18) and using the assumption on $\{\lambda_n\}$, these imply that

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|x_n - x^*\|^2 + \lambda_n^2\|A\|^2\|Sv_n - Ax_n\|^2 + 2\lambda_n\langle x_n - x^*, A^*(Sv_n - Ax_n)\rangle + \\
&\quad - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \\
&\leq \|x_n - x^*\|^2 + \lambda_n^2\|A\|^2\|Sv_n - Ax_n\|^2 \\
&\quad + 2\lambda_n\left(-\frac{1}{2}\|v_n - Ax_n\|^2 - \frac{1}{2}\|Sv_n - Ax_n\|^2\right) \\
&\quad - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \\
&= \|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 \\
&\quad - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \\
&\leq \|x_n - x^*\|^2 - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle. \qquad (22)
\end{aligned}$$

Now, we will show that

$$\langle\nabla f^{S\alpha_n}(x) - \nabla f^{S\alpha_n}(y), x - y\rangle \geq \frac{1}{\alpha_n + 2\|A\|^2}\|\nabla f^{S\alpha_n}(x) - \nabla f^{S\alpha_n}(y)\|^2. \qquad (23)$$

By Lemma 3, we have

$$\langle x - y, \nabla f^S(x) - \nabla f^S(y)\rangle \geq \frac{1}{2\|A\|^2}\|\nabla f^S(x) - \nabla f^S(y)\|^2.$$

Observe that

$$\begin{aligned}
&(\alpha_n + 2\|A\|^2)\langle\nabla f^{S\alpha_n}(x) - \nabla f^{S\alpha_n}(y), x - y\rangle \\
&\quad = (\alpha_n + 2\|A\|^2)\left(\alpha_n\|x - y\|^2 + \langle\nabla f^S(x) - \nabla f^S(y), x - y\rangle\right) \\
&\quad = \alpha_n^2\|x - y\|^2 + \alpha_n\langle\nabla f^S(x) - \nabla f^S(y), x - y\rangle + 2\alpha_n\|A\|^2\|x - y\|^2 \\
&\qquad + 2\|A\|^2\langle\nabla f^S(x) - \nabla f^S(y), x - y\rangle \\
&\quad \geq \alpha_n^2\|x - y\|^2 + \alpha_n\langle\nabla f^S(x) - \nabla f^S(y), x - y\rangle + 2\alpha_n\|A\|^2\|x - y\|^2 \\
&\qquad + \|\nabla f^S(x) - \nabla f^S(y)\|^2 \\
&\quad \geq \alpha_n^2\|x - y\|^2 + 2\alpha_n\langle\nabla f^S(x) - \nabla f^S(y), x - y\rangle + \|\nabla f^S(x) - \nabla f^S(y)\|^2 \\
&\quad = \|\alpha_n(x - y) + \nabla f^S(x) - \nabla f^S(y)\|^2 \\
&\quad = \|\nabla f^{S\alpha_n}(x) - \nabla f^{S\alpha_n}(y)\|^2.
\end{aligned}$$

By Proposition 1(2) one gets that

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|x_n - \lambda_n\nabla f^{S\alpha_n}(y_n) - x^*\|^2 - \|x_n - \lambda_n\nabla f^{S\alpha_n}(y_n) - z_n\|^2 \\
&= \|x_n - x^*\|^2 - 2\lambda_n\langle x_n - x^*, \nabla f^{S\alpha_n}(y_n)\rangle + \lambda_n^2\|\nabla f^{S\alpha_n}(y_n)\|^2 \\
&\quad - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, \nabla f^{S\alpha_n}(y_n)\rangle - \lambda_n^2\|\nabla f^{S\alpha_n}(y_n)\|^2 \\
&= \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle\nabla f^{S\alpha_n}(y_n), x^* - z_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - 2\lambda_n\Big(\langle\nabla f^{S\alpha_n}(y_n) - \nabla f^{S\alpha_n}(x^*), y_n - x^*\rangle \\
&\quad + \langle\nabla f^{S\alpha_n}(x^*), x^* - y_n\rangle + \langle\nabla f^{S\alpha_n}(y_n), y_n - z_n\rangle\Big) \\
&\leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle\nabla f^{S\alpha_n}(y_n), y_n - z_n\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\langle x_n - \lambda_n \nabla f^{S\alpha_n}(y_n) - y_n, z_n - y_n\rangle.
\end{aligned}
$$

Combining (23) with Proposition 1(1), we have

$$
\begin{aligned}
&\langle x_n - \lambda_n \nabla f^{S\alpha_n}(y_n) - y_n, z_n - y_n\rangle \\
&\quad = \langle x_n - \lambda_n \nabla f^{S\alpha_n}(x_n) - y_n, z_n - y_n\rangle + \lambda_n\langle \nabla f^{S\alpha_n}(x_n) - \nabla f^{S\alpha_n}(y_n), z_n - y_n\rangle \\
&\quad \le \lambda_n\langle \nabla f^{S\alpha_n}(x_n) - \nabla f^{S\alpha_n}(y_n), z_n - y_n\rangle \\
&\quad \le \lambda_n\|\nabla f^{S\alpha_n}(x_n) - \nabla f^{S\alpha_n}(y_n)\|\|z_n - y_n\| \\
&\quad \le \lambda_n(\alpha_n + 2\|A\|^2)\|x_n - y_n\|\|z_n - y_n\|.
\end{aligned} \tag{24}
$$

The hypothesis of $\{\lambda_n\}$ and (24), it follows that

$$
\begin{aligned}
\|z_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\langle x_n - \lambda_n \nabla f^{S\alpha_n}(y_n) - y_n, z_n - y_n\rangle \\
&\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\lambda_n(\alpha_n + 2\|A\|^2)\|x_n - y_n\|\|z_n - y_n\| \\
&\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + \|z_n - y_n\|^2 \\
&\quad + \lambda_n^2(\alpha_n + 2\|A\|^2)^2\|x_n - y_n\|^2 \\
&= \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2\|x_n - y_n\|^2 \\
&\le \|x_n - x^*\|^2.
\end{aligned} \tag{25}
$$

Likewise, we get that

$$
\|z_n - x^*\|^2 = \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2\|z_n - y_n\|. \tag{26}
$$

Since T is a pseudo-contractive mapping, we obtain that

$$
\|Tz_n - x^*\|^2 \le \|z_n - x^*\|^2 + \|z_n - Tz_n\|^2, \tag{27}
$$

and

$$
\begin{aligned}
\|Tw_n - x^*\|^2 &= \|T((1-\sigma_n)z_n + \sigma_n Tz_n) - x^*\|^2 \\
&\le \|(1-\sigma_n)(z_n - x^*) + \sigma_n(Tz_n - x^*)\|^2 \\
&\quad + \|(1-\sigma_n)z_n + \sigma_n Tz_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2.
\end{aligned} \tag{28}
$$

Again using (11) and T is an L-Lipschitzian pseudo-contractive mapping, these imply that

$$
\begin{aligned}
&\|(1-\sigma_n)z_n + \sigma_n Tz_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 \\
&\quad = \|(1-\sigma_n)(z_n - T((1-\sigma_n)z_n + \sigma_n Tz_n) + \sigma_n(Tz_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 \\
&\quad = (1-\sigma_n)\|z_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 + \sigma_n\|Tz_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 \\
&\qquad - \sigma_n(1-\sigma_n)\|z_n - Tz_n\|^2 \\
&\quad \le (1-\sigma_n)\|z_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 + \sigma_n L^2\|z_n - Tz_n\|^2 \\
&\qquad - \sigma_n(1-\sigma_n)\|z_n - Tz_n\|^2 \\
&\quad = (1-\sigma_n)\|z_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 - \sigma_n(1-\sigma_n - L^2)\|z_n - Tz_n\|^2.
\end{aligned} \tag{29}
$$

Combining with (11) and (27), we get that

$$\begin{aligned}
&\|(1-\sigma_n)(z_n-x^*)+\sigma_n(Tz_n-x^*)\|^2 \\
&\quad= (1-\sigma_n)\|z_n-x^*\|^2+\sigma_n\|Tz_n-x^*\|^2-\sigma_n(1-\sigma_n)\|z_n-Tz_n\|^2 \\
&\quad\le (1-\sigma_n)\|z_n-x^*\|^2+\sigma_n[\|z_n-x^*\|^2+\|z_n-Tz_n\|^2] \\
&\qquad- \sigma_n(1-\sigma_n)\|z_n-Tz_n\|^2 \\
&\quad= \|z_n-x^*\|^2+\sigma_n^2\|z_n-Tz_n\|^2.
\end{aligned} \tag{30}$$

By (29) and (30), it follows that

$$\begin{aligned}
\|Tw_n-x^*\|^2 &= \|T((1-\sigma_n)z_n+\sigma_nTz_n)-x^*\|^2 \\
&\le \|(1-\sigma_n)(z_n-x^*)+\sigma_n(Tz_n-x^*)\|^2 \\
&\quad+ \|(1-\sigma_n)z_n+\sigma_nTz_n-T((1-\sigma_n)z_n+\sigma_nTz_n)\|^2 \\
&= \|z_n-x^*\|^2+(1-\sigma_n)\|z_n-T((1-\sigma_n)z_n+\sigma_nTz_n)\|^2 \\
&\quad- \sigma_n(1-2\sigma_n-\sigma_n^2L^2)\|z_n-Tz_n\|^2.
\end{aligned} \tag{31}$$

Likewise, since T is a pseudo-contractive mapping, we get that

$$\|Ts_n-x^*\|^2 \le \|s_n-x^*\|^2+\|s_n-Ts_n\|^2. \tag{32}$$

Consider

$$\begin{aligned}
\|Ts_n-x^*\|^2 &= \|T((1-\beta_n)z_n+\beta_nTw_n)-x^*\|^2 \\
&\le \|(1-\beta_n)(z_n-x^*)+\beta_n(Tw_n-x^*)\|^2 \\
&\quad+ \|(1-\beta_n)z_n+\beta_nTw_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2,
\end{aligned} \tag{33}$$

and by combining with (11) and (31), we get that

$$\begin{aligned}
&\|(1-\beta_n)(z_n-x^*)+\beta_n(Tw_n-x^*)\|^2 \\
&\quad= (1-\beta_n)\|z_n-x^*\|^2+\beta_n\|Tw_n-x^*\|^2-\beta_n(1-\beta_n)\|z_n-Tw_n\|^2 \\
&\quad\le (1-\beta_n)\|z_n-x^*\|^2+\beta_n\left(\|z_n-x^*\|^2+(1-\sigma_n)\|z_n-Tw_n\|^2\right. \\
&\qquad\left.-\sigma_n(1-2\sigma_n-\sigma_n^2L^2)\|z_n-Tz_n\|^2\right)-\beta_n(1-\beta_n)\|z_n-Tw_n\|^2 \\
&\quad\le \|z_n-x^*\|^2-\beta_n(\sigma_n-\beta_n)\|z_n-Tw_n\|^2-\beta_n\sigma_n(1-2\sigma_n-\sigma_n^2L^2)\|z_n-Tz_n\|^2.
\end{aligned} \tag{34}$$

Again using (11) and T is an L-Lipschitzian pseudo-contractive mapping , we get that

$$\begin{aligned}
&\|(1-\beta_n)z_n+\beta_nTw_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2 \\
&\quad= \|(1-\beta_n)(z_n-T((1-\beta_n)z_n+\beta_nTw_n)+\beta_n(Tw_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2 \\
&\quad= (1-\beta_n)\|z_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2+\beta_n\|Tw_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2 \\
&\qquad- \beta_n(1-\beta_n)\|z_n-Tw_n\|^2 \\
&\quad= (1-\beta_n)\|z_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2+\beta_nL^2\|w_n-((1-\beta_n)z_n+\beta_nTw_n)\|^2 \\
&\qquad- \beta_n(1-\beta_n)\|z_n-Tw_n\|^2.
\end{aligned} \tag{35}$$

Since $w_n=(1-\sigma_n)z_n+\sigma_nTz_n$ and $\sigma_n<d<\frac{1}{\sqrt{L^2+1}+1+L^2}$, we have

$$\|w_n-(1-\beta_n)z_n-\beta_nTw_n\|^2 = \|(1-\sigma_n)z_n+\sigma_nTz_n-(1-\beta_n)z_n-\beta_nTw_n\|^2$$

$$
\begin{aligned}
&= \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2 \\
&\quad - 2\beta_n\sigma_n\langle z_n - Tw_n, z_n - Tz_n\rangle \\
&= \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2 \\
&\quad - 2\beta_n\sigma_n\langle z_n - Tw_n + Tz_n - Tz_n, z_n - Tz_n\rangle \\
&= \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2 \\
&\quad - 2\beta_n\sigma_n\|z_n - Tz_n\|^2 - 2\beta_n\sigma_n\langle Tz_n - Tw_n, z_n - Tz_n\rangle \\
&\leq \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2 \\
&\quad - 2\beta_n\sigma_n\|z_n - Tz_n\|^2 + 2\beta_n\sigma_n\|Tz_n - Tw_n\|\|Tz_n - z_n\| \\
&\leq \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2 \\
&\quad - 2\beta_n\sigma_n\|z_n - Tz_n\|^2 + 2\beta_n\sigma_n^2 L\|z_n - Tz_n\|\|Tz_n - z_n\| \\
&= \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2 \\
&\quad - 2\beta_n\sigma_n(1-\sigma_n L)\|z_n - Tz_n\|^2 \\
&\leq \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2. \qquad (36)
\end{aligned}
$$

Combining (35) with (36) obtain that

$$
\begin{aligned}
&\|(1-\beta_n)z_n + \beta_n Tw_n - T((1-\beta_n)z_n + \beta_n Tw_n)\|^2 \\
&\quad = (1-\beta_n)\|z_n - Ts_n\|^2 + \beta_n L^2\|w_n - ((1-\beta_n)z_n + \beta_n Tw_n)\|^2 \\
&\qquad - \beta_n(1-\beta_n)\|z_n - Tw_n\|^2 \\
&\quad \leq (1-\beta_n)\|z_n - Ts_n\|^2 + \beta_n L^2(\beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2) \\
&\qquad - \beta_n(1-\beta_n)\|z_n - Tw_n\|^2 \\
&\quad = (1-\beta_n)\|z_n - Ts_n\|^2 + \beta_n\sigma_n^2 L^2\|z_n - Tz_n\|^2 \\
&\qquad - \beta_n(1-\beta_n-\beta_n^2L^2)\|z_n - Tw_n\|^2. \qquad (37)
\end{aligned}
$$

From (33), (34) and (37), these imply that

$$
\begin{aligned}
\|Ts_n - x^*\|^2 &= \|T((1-\beta_n)z_n + \beta_n Tw_n) - x^*\|^2 \\
&\leq \|(1-\beta_n)(z_n - x^*) + \beta_n(Tw_n - x^*)\|^2 \\
&\quad + \|(1-\beta_n)z_n + \beta_n Tw_n - T((1-\beta_n)z_n + \beta_n Tw_n)\|^2 \\
&\leq \|z_n - x^*\|^2 - \beta_n(\sigma_n - \beta_n)\|z_n - Tw_n\|^2 \\
&\quad - \beta_n\sigma_n(1 - 2\sigma_n - \sigma_n^2L^2)\|z_n - Tz_n\|^2 \\
&\quad + (1-\beta_n)\|z_n - Ts_n\|^2 + \beta_n\sigma_n^2L^2\|z_n - Tz_n\|^2 \\
&\quad - \beta_n(1-\beta_n-\beta^2L^2)\|z_n - Tw_n\|^2 \\
&= \|z_n - x^*\|^2 + (1-\beta_n)\|z_n - Ts_n\|^2 \\
&\quad - \beta_n\left((\sigma_n - \beta_n) + (1-\beta_n-\beta_n^2L^2)\right)\|Tw_n - z_n\|^2 \\
&\quad - \beta_n\sigma_n(1 - \sigma_n(2+L^2) - \sigma_n^2L^2)\|z_n - Tz_n\|^2. \qquad (38)
\end{aligned}
$$

Since $\beta_n < c < \sigma_n < d < \frac{1}{\sqrt{L^2+1}+1+L^2}$, it obtains that

$$
1 - \beta_n - \beta_n^2L^2 > 0 \quad \text{and} \quad 1 - \sigma_n(2+L^2) - \sigma_n^2L^2 > 0.
$$

Therefore

$$\|Ts_n - x^*\|^2 \leq \|z_n - x^*\|^2 + (1-\beta_n)\|z_n - Ts_n\|^2. \tag{39}$$

From (11), (16) and (39), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\gamma_n)z_n + \gamma_n Ts_n - x^*\|^2 \\ &= (1-\gamma_n)\|z_n - x^*\|^2 + \gamma_n\|Ts_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|z_n - Ts_n\|^2 \\ &\leq (1-\gamma_n)\|z_n - x^*\|^2 + \gamma_n(\|z_n - x^*\|^2 + (1-\beta_n)\|z_n - Ts_n\|^2) \\ &\quad - \gamma_n(1-\gamma_n)\|z_n - Ts_n\|^2 \\ &= \|z_n - x^*\|^2 - \gamma_n(\beta_n - \gamma_n)\|z_n - Ts_n\|^2 \\ &\leq \|z_n - x^*\|^2. \end{aligned} \tag{40}$$

This together with (26) implies that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|,$$

for every $x^* \in \Gamma$ and for all $n \geq 0$. Therefore the sequence $\{x_n\}$ generated by algorithm (16) is Féjermonotone with respect to Γ. Thus we obtain $\lim_{n\to\infty} \|x_n - x^*\|$ exists immediately, it follows that $\{x_n\}$ is bounded and the sequence $\{\|x_n - x^*\|\}$ is monotonically decreasing. Moreover, $\{y_n\}$ and $\{z_n\}$ are also bounded sequences by using (22) and (25).

Combining (24) and (40), these imply that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2.\end{aligned}$$

It follows that

$$(1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

and so

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{41}$$

Likewise, we get

$$\lim_{n\to\infty} \|z_n - y_n\| = 0.$$

In combination (41), (22) and $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\begin{aligned}&\lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 + \lambda_n\|v_n - Ax^*\|^2 \\ &\quad\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \\ &\quad\leq (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\| - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|v_n - Ax_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Sv_n - Ax_n\| = 0.$$

So $\lim_{n\to\infty} \|v_n - Sv_n\| = 0$. From (40), we get that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2 - \gamma_n(\beta_n - \gamma_n)\|z_n - Ts_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \gamma_n(\beta_n - \gamma_n)\|z_n - Ts_n\|^2.\end{aligned}$$

It follows that

$$\gamma_n(\beta_n - \gamma_n)\|z_n - Ts_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

and so

$$\lim_{n\to\infty} \|z_n - Ts_n\| = 0.$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned}\|z_n - Tz_n\| &\le \|z_n - Ts_n\| + \|Ts_n - Tz_n\| \\ &\le \|z_n - Ts_n\| + L\|s_n - z_n\|.\end{aligned}$$

Since $s_n = (1 - \beta_n)z_n + \beta_n Tw_n$, we have

$$\begin{aligned}\|z_n - Tz_n\| &\le \|z_n - Ts_n\| + L\|s_n - z_n\| \\ &= \|z_n - Ts_n\| + \beta_n L\|z_n - Tw_n\|.\end{aligned} \tag{42}$$

Since $w_n = (1 - \sigma_n)z_n + \sigma_n Tz_n$, we get that

$$\begin{aligned}\|z_n - Tw_n\| &\le \|z_n - Ts_n\| + \|Ts_n - Tw_n\| \\ &\le \|z_n - Ts_n\| + L\|s_n - w_n\| \\ &= \|z_n - Ts_n\| + \sigma_n L\|z_n - Tz_n\| + \beta_n L\|z_n - Tw_n\|.\end{aligned} \tag{43}$$

So,

$$(1 - \beta_n L)\|Tw_n - z_n\| \le \|z_n - Ts_n\| + \sigma_n L\|z_n - Tz_n\|. \tag{44}$$

By (42) and (44), we get that

$$\begin{aligned}\|z_n - Tz_n\| &\le \|z_n - Ts_n\| + \beta_n L\|z_n - Tw_n\| \\ &= \|z_n - Ts_n\| + \beta_n L\left(\frac{1}{1 - \beta_n L}\|z_n - Ts_n\| + \frac{\sigma_n L}{1 - \beta_n L}\|z_n - Tz_n\|\right) \\ &= \left(1 + \frac{\beta_n L}{1 - \beta_n L}\right)\|z_n - Ts_n\| + \frac{\sigma_n \beta_n L^2}{1 - \beta_n L}\|z_n - Tz_n\|.\end{aligned}$$

This implies that

$$\|z_n - Tz_n\| \le \left(\frac{1}{1 - \beta_n L - \sigma_n \beta_n L^2}\right)\|z_n - Ts_n\|.$$

Therefore

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0.$$

By (44), we have

$$\lim_{n\to\infty} \|z_n - Tw_n\| = 0.$$

Using the firm nonexpansiveness of P_C, (9) and (22), we have

$$\begin{aligned}\|y_n - x^*\|^2 = \|P_C u_n - x^*\|^2 &\le \|u_n - x^*\|^2 - \|P_C u_n - u_n\|^2 \\ &\le \|x_n - x^*\|^2 - \|y_n - u_n\|^2.\end{aligned}$$

It follows that

$$\|y_n - u_n\|^2 \le \|x_n - x^*\|^2 - \|y_n - x^*\|^2$$

$$\leq (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\|.$$

From (41), we have

$$\lim_{n\to\infty} \|y_n - u_n\| = 0.$$

Since the sequence $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{x}$.

Therefore, from the above conclusions, we can obtain that

$$\begin{cases} x_{n_i} \rightharpoonup \hat{x}, \\ y_{n_i} \rightharpoonup \hat{x}, \\ u_{n_i} \rightharpoonup \hat{x}, \end{cases} \quad \text{and} \quad \begin{cases} z_{n_i} \rightharpoonup \hat{x}, \\ Ax_{n_i} \rightharpoonup A\hat{x}, \\ v_{n_i} \rightharpoonup A\hat{x}. \end{cases} \tag{45}$$

By Lemma 1, we have

$$\hat{x} \in F(T) \quad \text{and} \quad A\hat{x} \in F(S).$$

From $y_{n_i} = P_C u_{n_i} \in C$ and $v_{n_i} = P_Q Ax_{n_i}$ and by combining with (45), we get that

$$\hat{x} \in C \quad \text{and} \quad A\hat{x} \in Q.$$

Therefore

$$\hat{x} \in C \cap F(T) \quad \text{and} \quad A\hat{x} \in Q \cap F(S).$$

We can conclude that $\hat{x} \in \Gamma$ and this shows that $\omega_W(x_n) \subset \Gamma$. Since the $\lim_{n\to\infty} \|x_n - x^*\|$ exists for every $x^* \in \Gamma$ and every subsequence of $\{x_n\}$ converges weakly to $x^* \in \Gamma$, it is immediate from Lemma 2 that $\{x_n\}$ converges weakly to $x^* \in \Gamma$. This completes the proof. □

Next, utilizing Theorem 1, we give the following corollary using the iterative method by combining an extragradient method with regularization due to the Ishikawa iterative method.

Corollary 1 *Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\right), \\ z_n = P_C\left(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)y_n\right), \\ w_n = (1 - \sigma_n)z_n + \sigma_n T z_n, \\ x_{n+1} = (1 - \beta_n)z_n + \beta_n T w_n, \end{cases} \tag{46}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $0 < a < \beta_n < b < \sigma_n < c < \frac{1}{\sqrt{L^2+1}+1+L^2}$ Then the sequence $\{x_n\}$ generated by algorithm (46) converges weakly to an element of Γ.

Proof Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in F(T) \cap C$ and $Ax^* \in F(S) \cap Q$. Set $v_n = P_Q Ax_n$, $u_n = x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n$,

$\nabla f^{S\alpha_n} = A^*(I - SP_Q)A + \alpha_n I$ and $\nabla f^S = A^*(I - SP_Q)A$, for all $n \geq 0$. As in Theorem 1, we have

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \lambda_n \|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle, \tag{47}$$

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2\|x_n - y_n\|^2 \leq \|x_n - x^*\|^2, \tag{48}$$

$$\|z_n - x^*\|^2 = \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2\|z_n - y_n\|, \tag{49}$$

and

$$\|Tw_n - x^*\|^2 \leq \|z_n - x^*\|^2 + (1 - \sigma_n)\|z_n - T((1 - \sigma_n)z_n + \sigma_n Tz_n)\|^2 - \sigma_n(1 - 2\sigma_n - \sigma_n^2 L^2)\|z_n - Tz_n\|^2.$$

Since $b < \sigma_n < c < \frac{1}{\sqrt{L^2+1}+1+L^2}$, we obtain that

$$\|Tw_n - x^*\|^2 \leq \|z_n - x^*\|^2 + (1 - \sigma_n)\|z_n - T((1 - \sigma_n)z_n + \sigma_n Tz_n)\|^2. \tag{50}$$

From (47) and (50), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1 - \beta_n)z_n + \beta_n Tw_n - x^*\|^2 \\
&= (1 - \beta_n)\|z_n - x^*\|^2 + \beta_n\|Tw_n - x^*\|^2 - \beta_n(1 - \beta_n)\|z_n - Tw_n\|^2 \\
&\leq (1 - \beta_n)\|z_n - x^*\|^2 + \beta_n(\|z_n - x^*\|^2 + (1 - \sigma_n)\|z_n - Tw_n\|^2) \\
&\quad - \beta_n(1 - \beta_n)\|z_n - Tw_n\|^2 \\
&= \|z_n - x^*\|^2 - \beta_n(\sigma_n - \beta_n)\|z_n - Tw_n\|^2 \\
&\leq \|z_n - x^*\|^2 \leq \|x_n - x^*\|^2.
\end{aligned} \tag{51}$$

This implies that $\{x_n\}$ is a bounded sequence and the sequence $\{\|x_n - x^*\|\}$ is monotonically decreasing. Thus $\lim_{n\to\infty}\|x_n - x^*\|$ exists immediately, Moreover, $\{y_n\}$ and $\{z_n\}$ are also bounded sequences. In the same process of the proof in Theorem 1, we get that

$$\lim_{n\to\infty}\|z_n - y_n\| = \lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|y_n - u_n\| = 0,$$

and by (47), we obtain that

$$\lim_{n\to\infty}\|v_n - Ax_n\| = \lim_{n\to\infty}\|Sv_n - Ax_n\| = 0.$$

From (51), we observe that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2 - \beta_n(\sigma_n - \beta_n)\|z_n - Tw_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \beta_n(\sigma_n - \beta_n)\|z_n - Tw_n\|^2.
\end{aligned} \tag{52}$$

Thus

$$\beta_n(\sigma_n - \beta_n)\|z_n - Tw_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{53}$$

By taking the limit of $n \to \infty$, we have

$$\lim_{n\to\infty}\|z_n - Tw_n\| = 0.$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned}\|z_n - Tz_n\| &\leq \|z_n - Tw_n\| + \|Tw_n - Tz_n\| \\ &\leq \|z_n - Tw_n\| + L\|(1-\sigma_n)z_n + \sigma_n Tz_n - z_n\| \\ &\leq \|z_n - Tw_n\| + \sigma_n L\|z_n - Tz_n\|.\end{aligned}$$

It follows that

$$(1-\sigma_n L)\|z_n - Tz_n\| \leq \|z_n - Tw_n\|.$$

Therefore

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0.$$

Consequently, all conditions in Theorem 1 are satisfied and we can conclude that Corollary 1 can be obtained from Theorem 1 immediately. □

Next, utilizing Theorem 1, we give the following corollary when omit $\{z_n\}$ in the iterative method of Theorem 1.

Corollary 2 *Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\right), \\ w_n = (1-\sigma_n)y_n + \sigma_n Ty_n, \\ s_n = (1-\beta_n)y_n + \beta_n Tw_n, \\ x_{n+1} = (1-\gamma_n)y_n + \gamma_n Ts_n, \end{cases} \tag{54}$$

where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0,\infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $0 < a < \gamma_n < b < \beta_n < c < \sigma_n < d < \frac{1}{\sqrt{L^2+1}+1+L^2}$. Then the sequence $\{x_n\}$ generated by algorithm (54) converges weakly to an element of Γ.

Proof Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in F(T) \cap C$ and $Ax^* \cap F(S) \in Q$. Set $v_n = P_Q Ax_n$, $u_n = x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n$, $\nabla f^{S\alpha_n} = A^*(I - SP_Q)A + \alpha_n I$ and $\nabla f^S = A^*(I - SP_Q)A$, for all $n \geq 0$. As in Theorem 1, we have

$$\begin{aligned}\|y_n - x^*\|^2 \leq\ & \|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 \\ & - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle\end{aligned} \tag{55}$$

and

$$\|x_{n+1} - x^*\|^2 \leq \|y_n - x^*\|^2. \tag{56}$$

This implies that $\{x_n\}$ is a bounded sequence.

In view of (55) and (56), we obtain that

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|Sv_n - Ax_n\| = 0. \tag{57}$$

Therefore

$$\lim_{n\to\infty} \|v_n - Sv_n\| = 0.$$

Since P_C is firmly nonexpansive, we have

$$\begin{aligned}\|y_n - x^*\|^2 = \|P_C u_n - x^*\|^2 &\le \|u_n - x^*\|^2 - \|P_C u_n - u_n\|^2 \\ &\le \|x_n - x^*\|^2 - \|y_n - u_n\|^2.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{58}$$

By $u_n = x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n$ and (57), it follows that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0.$$

Combining with the previous equation and (58), we obtain that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

As in the proof of Theorem 1, we have $\lim_{n\to\infty} \|y_n - Ts_n\| = 0$. It follows that $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$.

Consequently, all conditions in Theorem 1 are satisfied and we can conclude that Corollary 2 can be obtained from Theorem 1 immediately. □

Next, utilizing Theorem 1, we illustrate the following corollary by setting $S : H_2 \to H_2$ to be an identity mapping in Theorem 1.

Corollary 3 *Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n(A^*(I - P_Q)A + \alpha_n I)x_n\right), \\ z_n = P_C\left(x_n - \lambda_n(A^*(I - P_Q)A + \alpha_n I)y_n\right), \\ w_n = (1 - \sigma_n)z_n + \sigma_n T z_n, \\ s_n = (1 - \beta_n)z_n + \beta_n T w_n, \\ x_{n+1} = (1 - \gamma_n)z_n + \gamma_n T s_n, \end{cases} \tag{59}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $0 \le \gamma_n < a < \beta_n < b < \sigma_n < c < \frac{1}{\sqrt{L^2+1}+1+L^2}$ Then the sequence $\{x_n\}$ generated by algorithm (59) converges weakly to an element of Γ.

Example 1 [26] Let H be the real Hilbert space $\mathbb{R}^2$ under the usual Euclidean inner product. If $x = (a, b) \in H$, define $x^\perp \in H$ to be $(b, -a)$. Let $K := \{x \in H : \|x\| \le 1\}$. and set

$$K_1 := \{x \in H : \|x\| \le \frac{1}{2}\} \quad \text{and} \quad K_2 := \{x \in H : \frac{1}{2} \le \|x\| \le 1\}.$$

Define $T : K \to K$ as follows:

$$Tx = \begin{cases} x + x^\perp & \text{if } x \in K_1, \\ \frac{x}{\|x\|} - x + x^\perp & \text{if } x \in K_2. \end{cases} \tag{60}$$

Then T is an L-Lipschitzian pseudo-contractive mapping with $L = 5$ and $F(T) = \{0\}$.

We next show that Example 1 satisfies all assumptions in Theorem 1 in order to illustrate the convergence of the sequence generated by the iterative process defined in Theorem 1 and compare its behavior with Ishikawa-type extragradient iterative method of Ceng et al. [25].

Example 2 Let $H_1 = H_2 = \mathbb{R}^2$ under the usual Euclidean inner product. Let $C = \{x \in H : \|x\| \leq 1\}$ and T as in Example 1. Let $Q = \mathbb{R}^2$ and $Sx = \frac{1}{3}x$ for all $x \in \mathbb{R}^2$. Set $Ax = \frac{1}{2}x$ for all $x \in \mathbb{R}^2$. Let $\lambda_n = \frac{n+1}{n+5}$, $\alpha_n = \frac{1}{(n+1)^2}$, $\sigma_n = 0.03$, $\beta_n = 0.025$, $\gamma_n = 0.01$ for all $n \in \mathbb{N}$. It is easy to see that $\Gamma = \{0\}$. Let $x_0 = (0.8, 0.6)$, then the sequence $\{x_n\}$ generated iteratively by (16) converges to 0 (Fig. 1; Table 1).

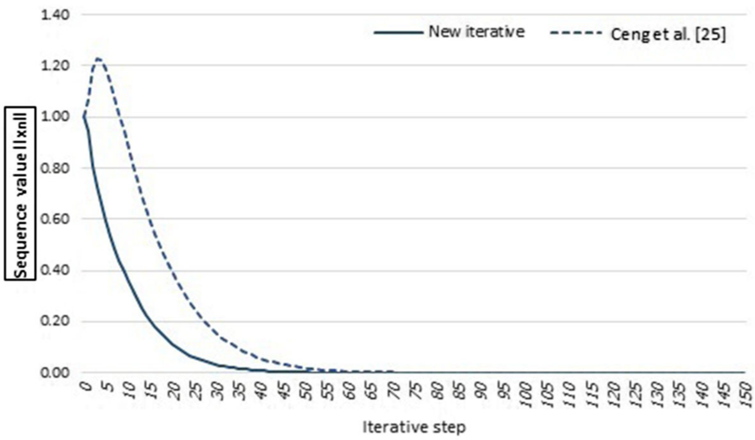

Fig. 1 The convergence of $\{x_n\}$ of Theorem 1 and Theorem 3.1 [25]

Table 1 The number of iterations for Example 2

No. of iterations	New iterative	Iterative of Ceng et al. [25]
0	(0.800000, 0.600000)	(0.800000, 0.600000)
10	(0.253946, 0.249351)	(0.633949, 0.605189)
20	(0.080841, 0.079395)	(0.284389, 0.271512)
⋮	⋮	⋮
80	(0.000035, 0.000034)	(0.000514, 0.000491)
90	(0.000009, 0.000009)	(0.000164, 0.000157)
91	(0.000008, 0.000008)	(0.000146, 0.000140)
⋮	⋮	⋮
110	(0.000001, 0.000001)	(0.000016, 0.000015)
111	(0.000001, 0.000001)	(0.000014, 0.000014)
112	(0.000000, 0.000000)	(0.000013, 0.000012)

3.2 The generalized Mann-type extragradient with regularization iterative method for pseudo-contractive mappings without Lipschitz assumption

Theorem 2 *Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and $T : C \to C$ be a continuous pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\right), \\ z_n = P_C\left(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)y_n\right), \\ x_{n+1} = \sigma_n z_n + \beta_n T z_n + \gamma_n x_n, \ \ n \geq 0, \end{cases} \tag{61}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $\{\gamma_n\}, \{\beta_n\}, \{\sigma_n\} \subset (a, b) \subset (0, 1)$ such that $\gamma_n + \beta_n + \sigma_n = 1$. Then the sequence $\{x_n\}$ generated by algorithm (61) converges weakly to an element of Γ.

Proof Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in F(T)\cap C$ and $Ax^* \in F(S)\cap Q$. Set $v_n = P_Q Ax_n, u_n = x_n - \lambda_n(A^*(I - SP_Q)Ax_n + \alpha_n I)x_n$, $\nabla f^{S\alpha_n} = A^*(I - SP_Q)A + \alpha_n I$ and $\nabla f^S = A^*(I - SP_Q)A$, for all $n \geq 0$. As in Theorem 1, we obtain that

$$\begin{aligned} \|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 \\ - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \end{aligned} \tag{62}$$

and

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2. \tag{63}$$

Likewise, we obtain that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|z_n - y_n\|^2.$$

In view of (11), (12), (62), and (63), this implies that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\gamma_n x_n + \sigma_n z_n + \beta_n T z_n - x^*\|^2 \\ &= \gamma_n\|x_n - x^*\|^2 + \sigma_n\|z_n - x^*\|^2 + \beta_n\|Tz_n - x^*\|^2 \\ &\quad - \gamma_n\sigma_n\|x_n - z_n\|^2 - \gamma_n\beta_n\|x_n - Tz_n\|^2 - \sigma_n\beta_n\|z_n - Tz_n\|^2 \\ &= \gamma_n\|x_n - x^*\|^2 + \sigma_n\|z_n - x^*\|^2 + \beta_n(\langle Tz_n - z_n, Tz_n - x^*\rangle \\ &\quad + \langle z_n - x^*, Tz_n - x^*\rangle) - \gamma_n\sigma_n\|x_n - z_n\|^2 - \gamma_n\beta_n\|x_n - Tz_n\|^2 \\ &\quad - \sigma_n\beta_n\|z_n - Tz_n\|^2 \\ &\leq \gamma_n\|x_n - x^*\|^2 + (\sigma_n + \beta_n)\|z_n - x^*\|^2 - \gamma_n\sigma_n\|x_n - z_n\|^2 \\ &\quad - \gamma_n\beta_n\|x_n - Tz_n\|^2 - \sigma_n\beta_n\|z_n - Tz_n\|^2 \\ &\leq \gamma_n\|x_n - x^*\|^2 + (\sigma_n + \beta_n)(\|x_n - x^*\|^2 \\ &\quad - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2) \\ &\quad - \gamma_n\sigma_n\|x_n - z_n\|^2 - \gamma_n\beta_n\|x_n - Tz_n\|^2 - \sigma_n\beta_n\|z_n - Tz_n\|^2 \\ &\leq \|x_n - x^*\|^2 - (\sigma_n + \beta_n)(1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2 \end{aligned}$$

$$- \gamma_n\sigma_n\|x_n - z_n\|^2 - \gamma_n\beta_n\|x_n - Tz_n\|^2 - \sigma_n\beta_n\|z_n - Tz_n\|^2. \qquad (64)$$

By the hypothesis of $\{\lambda_n\}$, we have

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|.$$

This implies that $\{\|x_n - x^*\|\}$ is a nonincreasing sequence and obtain that the limit of the sequence $\{\|x_n - x^*\|\}$ exists, we get that $\{x_n\}$ is a bounded sequence. From (64), we have

$$\begin{aligned}
&(\sigma_n + \beta_n)(1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2 + \gamma_n\sigma_n\|x_n - z_n\|^2 \\
&\quad + \gamma_n\beta_n\|x_n - Tz_n\|^2 + \sigma_n\beta_n\|z_n - Tz_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.
\end{aligned}$$

By the hypothesis of the parameters σ_n, β_n and γ_n , we obtain that

$$\lim_{n\to\infty}\|x_n - z_n\| = \lim_{n\to\infty}\|x_n - Tz_n\| = \lim_{n\to\infty}\|z_n - Tz_n\| = 0,$$

and

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \qquad (65)$$

Likewise, we have

$$\lim_{n\to\infty}\|z_n - y_n\| = 0.$$

Combining with (62), this implies that

$$\begin{aligned}
&\lambda_n\|v_n - Ax_n\|^2 + \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 \\
&\quad \leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \\
&\quad \leq (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\| - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle.
\end{aligned}$$

By the hypothesis of $\{\alpha_n\}$, $\{\lambda_n\}$ and (65), it follows that

$$\lim_{n\to\infty}\|v_n - Ax_n\| = \lim_{n\to\infty}\|Sv_n - Ax_n\| = 0.$$

Therefore,

$$\lim_{n\to\infty}\|v_n - Sv_n\| = 0.$$

As in the proof of Theorem 1, we get that

$$\lim_{n\to\infty}\|y_n - u_n\| = \lim_{n\to\infty}\|u_n - x_n\| = 0.$$

Consequently, all conditions in Theorem 1 are satisfied and we can conclude that Theorem 2 can be obtained immediately. □

Similarly as previous subsection, utilizing Theorem 2, we give the following corollary when changing the generalized Mann-type iterative method is the Mann-type iterative method

Corollary 4 *Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with* *its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and $T : C \to C$ be a continuous pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\right), \\ z_n = P_C\left(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)y_n\right), \\ x_{n+1} = (1-\beta_n)z_n + \beta_n T z_n, \ \ n \geq 0, \end{cases} \tag{66}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $\{\beta_n\} \subset (0, 1)$ such that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then the sequence $\{x_n\}$ generated by algorithm (66) converges weakly to an element of Γ.

Proof Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in F(T) \cap C$ and $Ax^* \in F(S) \cap Q$. Set $v_n = P_Q Ax_n$, $u_n = x_n - \lambda_n(A^*(I - SP_Q)Ax_n + \alpha_n I)x_n$, $\nabla f^{S\alpha_n} = A^*(I - SP_Q)A + \alpha_n I$ and $\nabla f^S = A^*(I - SP_Q)A$, for all $n \geq 0$. As in Theorem 1, we have

$$\begin{aligned} \|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 \\ - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \end{aligned} \tag{67}$$

and

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2. \tag{68}$$

Likewise, we obtain that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|z_n - y_n\|^2.$$

By (13) and (68), it follows that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\beta_n)z_n - \beta_n T z_n - x^*\|^2 \\ &= (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|Tz_n - x^*\|^2 - \beta_n(1-\beta_n)\|z_n - Tz_n\|^2 \\ &= (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\langle Tz_n - z_n, Tz_n - x^*\rangle \\ &\quad + \beta_n\langle z_n - x^*, Tz_n - x^*\rangle - \beta_n(1-\beta_n)\|z_n - Tz_n\|^2 \\ &\leq \|z_n - x^*\|^2 - \beta_n(1-\beta_n)\|z_n - Tz_n\|^2 \\ &\leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2 \\ &\quad - \beta_n(1-\beta_n)\|z_n - Tz_n\|^2. \end{aligned} \tag{69}$$

Therefore

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|.$$

Similarly, by the process in Theorem 2, we have $\{x_n\}$ is a bounded sequence. From (69), we have

$$\begin{aligned} (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2 + \beta_n(1-\beta_n)\|z_n - Tz_n\|^2 \\ \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \end{aligned} \tag{70}$$

It follows that

$$\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|z_n - Tz_n\| = 0.$$

Similarly, we have

$$\lim_{n\to\infty} \|z_n - y_n\| = 0.$$

As the same argument of Theorem 2, we get that

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|Sv_n - Ax_n\| = \lim_{n\to\infty} \|v_n - Sv_n\| = 0.$$

and

$$\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|u_n - y_n\| = 0.$$

Consequently, all conditions in Theorem 2 are satisfied and we can conclude that Corollary 4 can be obtained immediately. □

Next, utilizing Theorem 2, we give the following corollary when omit $\{z_n\}$ in the iterative method of Theorem 2.

Corollary 5 *Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and $T : C \to C$ be a continuous pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\right), \\ x_{n+1} = \sigma_n y_n + \beta_n T y_n + \gamma_n x_n, \ \ n \geq 0, \end{cases} \tag{71}$$

$\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $\{\gamma_n\}, \{\beta_n\}, \{\sigma_n\} \subset (a, b) \subset (0, 1)$ such that $\gamma_n + \beta_n + \sigma_n = 1$. Then the sequence $\{x_n\}$ generated by algorithm (71) converges weakly to an element of Γ.

Proof Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in C \cap F(T)$ and $Ax^* \in Q \cap F(S)$. Set $v_n = P_Q Ax_n, u_n = x_n - \lambda_n(A^*(I - SP_Q)Ax_n + \lambda_n I)\alpha_n x_n$ for all $n \geq 0$. As in Theorem 2, we have

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0$$

and

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|Sv_n - Ax_n\| = \lim_{n\to\infty} \|v_n - Sv_n\| = 0.$$

Similarly to Corollary 2, we obtain that

$$\lim_{n\to\infty} \|u_n - y_n\| = \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Consequently, all conditions in Theorem 2 are satisfied and we can conclude that Corollary 5 can be obtained immediately. □

Next, utilizing Theorem 2, we give the following corollary when define $S : H_2 \to H_2$ to be identity mapping in Theorem 2.

Corollary 6 *Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $T : C \to C$ be a continuous pseudo-contractive* *mapping such that $\Gamma \cap F(T) \neq \emptyset$. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n(A^*(I-P_Q)A + \alpha_n I)x_n\right), \\ z_n = P_C\left(x_n - \lambda_n(A^*(I-P_Q)A + \alpha_n I)y_n\right), \\ x_{n+1} = \alpha_n z_n + \beta_n T z_n + \gamma_n x_n, \ \ n \geq 0, \end{cases} \tag{72}$$

where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0,\infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $\{\gamma_n\}, \{\beta_n\}, \{\sigma_n\} \subset (a,b) \subset (0,1)$ such that $\gamma_n + \beta_n + \sigma_n = 1$. Then the sequence $\{x_n\}$ generated by algorithm (72) converges weakly to an element of Γ.

Next, we give the numerical example which satisfies all assumptions in Theorem 2 in order to illustrate the convergence of the sequence generated by the iterative process defined in Theorem 2 and compare its behavior with Mann-type extragradient iterative method of Ceng et al. [25].

Example 3 Let $H_1 = H_2 = \mathbb{R}$. Let $C = \mathbb{R}/\{-1\}$ and $Tx = -\frac{x}{(1+x)}$ for all $x \in C$. Since $\|Tx - Ty\|^2 \leq \|\frac{x-y}{(1+x)(1+y)}\|^2 \leq \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2$, then T is a continuous pseudocontractive mapping. Let $Q = \mathbb{R}$ and $Sx = \frac{1}{3}x$ for all $x \in \mathbb{R}$. Set $Ax = \frac{1}{2}x$ for all

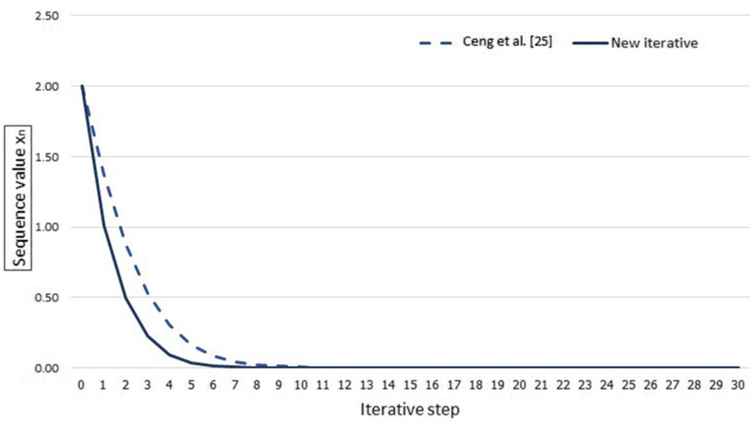

Fig. 2 The convergence of $\{x_n\}$ of Theorem 2 and Theorem 4.1 [25]

Table 2 The number of iterations for Example 3

No. of iterations	New iterative	Ceng et al. [25]
0	2.000000	2.000000
1	1.007543	1.369996
5	0.037257	0.164183
10	0000267	0.005400
⋮	⋮	.
14	0.000005	0.000304
15	0.000002	0.000147
16	0.000001	0.000071
17	0.000000	0.000034

$x \in \mathbb{R}$. Let $\lambda_n = \frac{n+1}{n+5}$, $\alpha_n = \frac{1}{(n+1)^2}$, $\sigma_n = 0.6$ $\beta_n = 0.3$, $\gamma_n = 0.1$ for all $n \in \mathbb{N}$. It is easy to see that $\Gamma = \{0\}$. Let the sequence $\{x_n\}$ be generated iteratively by (61), then the sequence $\{x_n\}$ converges to 0 (Fig. 2; Table 2).

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