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Term rank preservers of bisymmetric matrices over semirings

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Abstract: In this article, we introduce another standard form of linear preservers. This new standard form provides characterizations of the linear transformations on the set of bisymmetric matrices with zero diagonal and zero antidiagonal over antinegative semirings without zero divisors which preserve some sort of term ranks and preserve the matrix that can be determined as the greatest one. The numbers of all possible linear transformations satisfying each condition are also obtained.

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1. Introduction

Linear preserver problems (LPPs) are one of the most active research topics in matrix theory during the past half-century, which have been studied when linear transformations on spaces of matrices leave certain conditions invariant. An excellent reference for LPPs is Pierce et al. (1992). There are many works on LPPs over various algebraic structures. The spaces of matrices over semirings also have been one of them.

Rank preserver problems play a pivotal role in investigating questions regarding other preservers. It can be found in Young & Choi (2008) that there are many nonequivalent definitions of rank

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PUBLIC INTEREST STATEMENT

A Boolean matrix is a matrix whose entries are 0 or 1 with addition and multiplication defined as if 0 and 1 are integers, except that 1 + 1 = 1. The term rank of a Boolean matrix A is the minimum number of lines containing all nonzero entries of A. Recently, the study of linear transformations preserving some certain functions over sets of many kinds of matrices has become established. In this article, we provide the patterns of linear transformations preserving term ranks of bisymmetric Boolean matrices via our new standard form. The number of such linear preservers is also counted. Moreover, the results on bisymmetric Boolean matrices are extended to bisymmetric matrices over semirings. We believe that the standard form that we have cooked up will be beneficial and will give a new point of view for research in related areas.

functions for matrices over semirings. Among many essential different definitions of rank functions of matrices over semirings, the combinatorial approach leads to the *term ranks* of such matrices.

Inspired by Beasley, Song and Kang's recently work, in Beasley, Song, & Kang (2012), on term rank preservers of symmetric matrices with zero diagonal over commutative antinegative semirings with no zero divisors, we investigate linear transformations on bisymmetric matrices whose all diagonal and antidiagonal entries are zeroes over such semirings that preserve some term ranks. We refer to Zhao, Hu, & Zhang (2008) and the references therein for more results and applications of bisymmetric matrices.

The significant survey about LPPs in Chapter 22 of Hogben (2007) indicates that linear preservers often have the standard forms. It turns out that our linear preservers do not possess any former standard forms; however, we invent a new standard form in order to obtain a natural and intrinsic characterization of term rank preserver on bisymmetric matrices whose all diagonal and antidiagonal entries are zeroes over commutative antinegative semirings with no zero divisors.

We organize this article as follows. In Section 2, some of the well-known terminologies and results on LPPs are reviewed and the notations in our work are introduced. In the third section, the results on term rank preservers of bisymmetric Boolean matrices with zero diagonal and zero antidiagonal are presented. Then, we extend the results to the case that all entries of such matrices are in commutative antinegative semirings with no zero divisors in the last section.

2. Definitions and preliminaries

We begin this section with the definition of a semiring. See Golan (1999) for more information about semirings and their properties.

Definition 2.1. A *semiring* $(\mathscr{S}, +, \cdot)$ is a set $\mathscr{S}$ with two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that:

(i). $(\mathscr{S}, +)$ is a commutative monoid (the identity is denoted by 0);
(ii). $(\mathscr{S}, \cdot)$ is a semigroup (the identity, if exists, is denoted by 1);
(iii). multiplication is distributive over addition on both sides;
(iv). $s \cdot 0 = 0 \cdot s = 0$ for all $s \in S$.

We say that $(\mathscr{S}, +, \cdot)$ is a *commutative semiring* if $(\mathscr{S}, \cdot)$ is a commutative semigroup. A semiring is *antinegative* if the only element having an additive inverse is the additive identity. For a commutative semiring $\mathscr{S}$, a nonzero element $s \in \mathscr{S}$ is called a *zero divisor* if there exists a nonzero element $t \in \mathscr{S}$ such that $s \cdot t = 0$.

Throughout this article, we let $\mathscr{S}$ be a commutative antinegative semiring containing the multiplicative identity with no zero divisors and $\cdot$ is denoted by juxtaposition.

One of the simplest examples of an antinegative semiring without zero divisors is the *binary Boolean algebra* $\mathscr{B}$ which is a set of only two elements 0 and 1 with addition and multiplication on $\mathscr{B}$ defined as though 0 and 1 were integers, except that $1 + 1 = 1$. Note that $\mathscr{B}$ is also called a *Boolean semiring*. Another example of these semirings is the *fuzzy semiring* which is the real interval $[0, 1]$ with the maximum and minimum as its addition and multiplication, respectively. Besides, any nonnegative subsemirings of $\mathbb{R}$ are other examples of such semirings.

Let $M_{m,n}(\mathscr{S})$ denote the set of all $m \times n$ matrices over $\mathscr{S}$. The usual definitions for addition and multiplication of matrices are applied to such matrices as well. The notation $M_n(\mathscr{S})$ is used when $m = n$. A matrix in $M_{m,n}(\mathscr{B})$ is called a *Boolean matrix*. A square matrix obtained by permuting rows of the identity matrix is called a *permutation matrix*.

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In order to investigate LPPs on $M_{m,n}(\mathscr{S})$, we give the notions of a linear transformation on $M_{m,n}(\mathscr{S})$ and when it preserves some certain properties.

Definition 2.2. A mapping $T : M_{m,n}(\mathscr{S}) \to \mathscr{M}_{m,n}(\mathscr{S})$ is said to be a *linear transformation if* $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$ for all $X, Y \in M_{m,n}(\mathscr{S})$ and $\alpha, \beta \in \mathscr{S}$.

Let $\mathscr{K}$ be a subset of $M_{m,n}(\mathscr{S})$ containing all matrices with a property P and T a linear transformation on $M_{m,n}(\mathscr{S})$. Then, we say that T *preserves the property* P if $T(X) \in \mathscr{K}$ whenever $X \in \mathscr{K}$ for all $X \in M_{m,n}(\mathscr{S})$ and T *strongly preserves the property* P if $T(X) \in \mathscr{K}$ if and only if $X \in \mathscr{K}$ for all $X \in M_{m,n}(\mathscr{S})$.

We next state the most common concept of the standard form in the theory of linear preservers over semirings by the following definitions. Let $\mathbb{N}_n = \{1, 2, \ldots, n\}$.

Definition 2.3. A mapping $\sigma : \mathbb{N}_n \to \mathbb{N}_n$ is said to be a *half-mapping* (on $\mathbb{N}_n$) if σ satisfies the following properties:

(i) $\sigma|_{\mathbb{N}_{\lceil \frac{n}{2} \rceil}}$ is a permutation on $\mathbb{N}_{\lceil \frac{n}{2} \rceil}$, and

(ii) $\sigma(n - i + 1) = n - \sigma(i) + 1$ for all $i \in \mathbb{N}_{\lfloor \frac{n}{2} \rfloor}$.

If a half-mapping is also a permutation on $\mathbb{N}_n$, then we call it a *half-permutation*.

Note that a half-mapping σ on $\mathbb{N}_n$ is a permutation if n is even, but σ is not necessarily a permutation when n is odd. If n is odd, and the half-mapping σ on $\mathbb{N}_n$ satisfies $\sigma(\lceil \frac{n}{2} \rceil) = \lceil \frac{n}{2} \rceil$, then σ is a permutation and hence a half-permutation.

Definition 2.4. Let e_i be an $n \times 1$ Boolean matrix whose only nonzero entry is in the ith row, σ a half-mapping on $\mathbb{N}_n$. We define the *matrix induced by* σ to be the $n \times n$ Boolean matrix whose the ith column is $e_{\sigma(i)}$ if $i \neq \lceil \frac{n}{2} \rceil$ and the $\lceil \frac{n}{2} \rceil$th column is $e_{\sigma(\lceil \frac{n}{2} \rceil)}$ when n is even, and $e_{\sigma(\lceil \frac{n}{2} \rceil)} + e_{n-\sigma(\lceil \frac{n}{2} \rceil)+1}$ when n is odd. The matrix induced by σ is denoted by P_σ.

Definition 2.5. Let $T : M_{m,n}(\mathscr{S}) \to M_{m,n}(\mathscr{S})$ be a linear transformation. We say that T *is induced by* (σ, τ, B) if there exist mappings σ and τ on $\mathbb{N}_m$ and $\mathbb{N}_n$, respectively, and a matrix $B \in M_{m,n}(\mathscr{S})$ such that either

(i) $T(X) = P_\sigma(X \circ B)Q_\tau$ for all $X \in M_{m,n}(\mathscr{S})$; or
(ii) $m = n$ and $T(X) = P_\sigma(X^t \circ B)Q_\tau$ for all $X \in M_{m,n}(\mathscr{S})$,

where $\circ$ denotes the *Schur product*, i.e., $A \circ B = [a_{ij}b_{ij}]$ for all $A = [a_{ij}], B = [b_{ij}] \in M_{m,n}(\mathscr{S})$.

We let $J_{m,n} \in M_{m,n}(\mathscr{S})$ denote the $m \times n$ matrix whose entries equal to 1. From now on, the subscripts of matrices may be dropped when the sizes of matrices are clear from the context. We simply say that T *is induced by* (σ, τ) when $B = J$. We also say that T *is induced by* σ if T is induced by (σ, σ^{-1}) where σ is a permutation on $\mathbb{N}_n$. In general (see (Beasley & Pullman, 1987)), a linear transformation T on $M_{m,n}(\mathscr{S})$ induced by (σ, τ, B) where σ and τ are permutations on $\mathbb{N}_m$ and $\mathbb{N}_n$, respectively, and all entries of B are nonzero, is usually called a (P, Q, B) *operator*, where P and Q are permutation matrices induced by σ and τ, respectively.

The characterizations of linear transformations preserving term ranks were first studied by Beasley and Pullman (Beasley & Pullman, 1987) in 1987. This work is continued later on (see (Kang & Song, 2012; Song & Beasley, 2013) and the references therein). We recall the definition of the term rank and its preserver theorem as follows.

Definition 2.6. The *term rank of* $A \in M_{m,n}(\mathscr{S})$ is the minimum number of rows or columns required to contain all the nonzero entries of A.

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Theorem 2.7. (Beasley & Pullman, 1987) *Let $T : M_{m,n}(\mathscr{S}) \to M_{m,n}(\mathscr{S})$ be a linear transformation. Then T preserves term ranks 1 and 2 if and only if T is induced by (σ, τ, B) where σ and τ are permutations on $\mathbb{N}_m$ and $\mathbb{N}_n$, respectively, and all entries of B are nonzero.*

Let $S_n^{(0)}(\mathscr{S})$ denote the set of all $n \times n$ symmetric matrices with entries in $\mathscr{S}$ and all diagonal entries equal 0. Let $A = [a_{ij}], B = [b_{ij}] \in S_n^{(0)}(\mathscr{S})$. The matrix A is said to *dominate* the matrix B, written $A \geq B$ or $B \leq A$, if $b_{ij} = 0$ whenever $a_{ij} = 0$ for all i, j. For $A \geq B$, the matrix $[x_{ij}]$ with $x_{ij} = a_{ij}$ if $b_{ij} = 0$ and $x_{ij} = 0$ otherwise, is denoted $A \backslash B$. Let I be the $n \times n$ identity matrix and $K = J \backslash I$. The following theorem is a characterization of linear transformations on $S_n^{(0)}(\mathscr{B})$ preserving some term ranks provided by Beasley and his colleagues (Beasley et al., 2012). They also generalized this result to $S_n^{(0)}(\mathscr{S})$.

Theorem 2.8. (Beasley et al., 2012) *Let $T : S_n^{(0)}(\mathscr{B}) \to S_n^{(0)}(\mathscr{B})$ be a linear transformation. Then T preserves term rank 2 and $T(K) = K$ if and only if T is induced by σ where σ is a permutation on $\mathbb{N}_n$.*

Recently, the study of linear transformations preserving a certain matrix function on $S_n^{(0)}(\mathscr{S})$, called the *star cover number*, was provided in Beasley, Song, Kang, & Lee (2013). The matrix in $S_n^{(0)}(\mathscr{B})$ whose entries in the ith row and the ith column, except at the (i, i) position, are 1 and 0 elsewhere is called the *full star* on row and column i. For $A \in S_n^{(0)}(\mathscr{S})$, a *star cover* of A is a sum of full stars dominating A and the *star cover number* of A is the minimum number of full stars whose sum dominates A.

Theorem 2.9. (Beasley et al., 2013) *Let $T : S_n^{(0)}(\mathscr{B}) \to S_n^{(0)}(\mathscr{B})$ be a linear transformation. Then the following are equivalent:*

(i) T preserves the star cover number 1 and $T(K) = K$;
(ii) T preserves the star cover numbers 1 and 2;
(iii) T is induced by σ where σ is a permutation on $\mathbb{N}_n$.

To investigate further on bisymmetric matrices over semirings $\mathscr{S}$, we let $BS_n^{(0)}(\mathscr{S})$ denote the set of all $n \times n$ bisymmetric matrices with entries in $\mathscr{S}$ and all diagonal and antidiagonal entries are 0. Note that if $A = [a_{ij}] \in BS_n^{(0)}(\mathscr{S})$, then $a_{i,i} = 0 = a_{i,n-i+1}$ and $a_{ij} = a_{j,i} = a_{n-j+1,n-i+1} = a_{n-i+1,n-j+1}$ for all $i \neq j$. For $i \neq j$, let $Q_{ij} \in BS_n^{(0)}(\mathscr{S})$ be the matrix whose the four entries at the (i,j), (j,i), $(n-j+1, n-i+1)$ and $(n-i+1, n-j+1)$ positions are 1 and other entries are 0. The matrix Q_{ij} is called a *quadrilateral cell* and the matrix $Q_{i,n-j+1}$ is called the *corresponding quadrilateral cell* of Q_{ij}. We let Ω_n (or Ω when the size of matrices is clear) denote the set of all quadrilateral cells in $BS_n^{(0)}(\mathscr{S})$. For each $i \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$, the sum of all Q_{ij} such that $j \notin \{i, n-i+1\}$ is called the *full double star* on rows and columns i and $n-i+1$, denoted by DS_i. A *double-star matrix* is a nonzero matrix in $BS_n^{(0)}(\mathscr{S})$ dominated by a full double star. Let $L = DS_1 + \cdots + DS_{\lceil \frac{n}{2} \rceil}$. The following notion is given in order to investigate more on the preserver problems of the matrix function on $BS_n^{(0)}(\mathscr{S})$ resembling the star cover number of matrices on $S_n^{(0)}(\mathscr{S})$.

Definition 2.10. Let $A \in BS_n^{(0)}(\mathscr{S})$. *A double-star cover of A is a sum of full double stars that dominates A. The double-star cover number of A is the minimum number of full double stars whose sum dominates A.*

Next, a generalization of a linear transformation on $BS_n^{(0)}(\mathscr{S})$ induced by (σ, σ^{-1}, B), which turns out to be the standard form for our results, is given by the following definition. We begin with the notation of certain square Boolean matrices. For each $A \in M_n(\mathscr{B})$, we let $A^{(0)} = A$ and $A^{(1)} = A \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}$.

Definition 2.11. Let $T : BS_n^{(0)}(\mathscr{B}) \to BS_n^{(0)}(\mathscr{B})$ be a linear transformation. We say that T *is induced by* (σ, F, G) if there exist a mapping σ on $\mathbb{N}_n$ and matrices $F = [f_{i,j}], G = [g_{i,j}] \in M_n(\mathscr{B})$ such that

$$T(A) = \sum_{Q_{i,j} \in \Omega_A} P_\sigma Q_{i,j} \left(P_\sigma^{(f_{i,j})} + P_\sigma^{(g_{i,j})} \right)^t$$

for all $A \in BS_n^{(0)}(\mathscr{B})$ where Ω_A is the set of all quadrilateral cells summing to A.

We say that T *is induced by* (σ, F) if T is induced by (σ, F, G) and $F = G$. Observe that (i) $P_\sigma Q_{i,j} \left(P_\sigma^{(0)}\right)^t = Q_{\sigma(i),\sigma(j)}$ and (ii) $P_\sigma Q_{i,j} \left(P_\sigma^{(1)}\right)^t = Q_{\sigma(i),n-\sigma(j)+1}$.

To illustrate more clearly, we give the following example.

Example 2.12. Let σ be the half-mapping on $\mathbb{N}_7$ defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}. \text{ Then } P_\sigma = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{Let } F = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and T be a linear transformation on $BS_7^{(0)}(\mathscr{B})$ induced by (σ, F, G). Then the following table shows the images of each quadrilateral cell.

We end this section by introducing some instrumental notions.

A	$Q_{1,2}$	$Q_{1,3}$	$Q_{1,4}$	$Q_{1,5}$	$Q_{1,6}$	$Q_{2,3}$	$Q_{2,4}$	$Q_{2,5}$	$Q_{3,4}$
$T(A)$	$Q_{3,4}$	$Q_{2,4}$	$Q_{1,4}$	$Q_{2,4}$	$Q_{3,4}$	$Q_{2,5} + Q_{2,3}$	$Q_{1,3} + Q_{1,5}$	$Q_{2,5}$	$Q_{1,6}$

Note that $T(L) \neq L$ and T does not preserve term rank 4 since $T(Q_{1,2}) = Q_{3,4}$.

Definition 2.13. Let $i, j \in \mathbb{N}_n$ such that $i \neq j$, $F = [f_{i,j}]$ and $G = [g_{i,j}]$ be Boolean matrices. We define the set

$$\Lambda_{ij} = \{(i,j), (j,i), (n-j+1, n-i+1), (n-i+1, n-j+1),$$

$$(i, n-j+1), (n-j+1, i), (j, n-i+1), (n-i+1, j)\}$$

and the set $F_{\Lambda_{ij}} = \{f_{p,q} \mid (p,q) \in \Lambda_{ij}\}$. We write $F \leq_\Lambda G$ if $G_{\Lambda_{ij}} = \{1\}$ whenever $1 \in F_{\Lambda_{ij}}$ for all $i \neq j$. We say that matrix F is *strongly dominated by* G, denoted by $F <_s G$, if there exists $i \in \mathbb{N}_n$ such that $F_{\Lambda_{ij}} = \{0\}$ and $G_{\Lambda_{ij}} = \{1\}$ for all $j \neq i$.

A matrix $A = [a_{i,j}] \in BS_n^{(0)}(\mathscr{S})$ is called a *tetrasymmetric* matrix if $a_{p,q} = a_{i,j}$ for all $(p,q) \in \Lambda_{i,j}$.

3. Term rank preservers of bisymmetric matrices over Boolean semirings

In this section, we first consider linear preservers of bisymmetric matrices with zero diagonal and zero antidiagonal over Boolean semirings in order to generalize this notions to such matrices over other semirings. The following observations are obtained from the structure of the quadrilateral cells and the proofs are skipped.

Lemma 3.1. *Let* $A \in BS_n^{(0)}(\mathscr{B})$.

(i) A *is of term rank* 2 *or* 4 *if and only if* A *is a double-star matrix.*

(ii) For each two distinct elements $i,j \in \mathbb{N}_n$, $A \leq DS_i$ *and* $A \leq DS_j$ *if and only if* $A \leq Q_{i,j} + Q_{i,n-j+1}$.

(iii) If the matrix L *can be written as* $L = F_1 + F_2 + \cdots + F_t$ *where each* F_i *is a full double-star matrix, then* $t \geq \lceil \frac{n}{2} \rceil - 1$.

The study of linear transformations that strongly preserve a certain property is a common research focus (see (Beasley & Song, 2016a, 2016b) and references therein). We provide a characterization of linear transformations on $BS_n^{(0)}(\mathscr{B})$ preserving the matrix L strongly; i.e., L is the only matrix mapped to itself.

Lemma 3.2. *Let* $T : BS_n^{(0)}(\mathscr{B}) \to BS_n^{(0)}(\mathscr{B})$ *be a linear transformation. Then* T *preserves the set of all nonzero matrices and* T *strongly preserves the matrix* L *if and only if* T *is bijective on* Ω.

Proof. The sufficient part is obvious. To show the necessary part, we first show that T is injective on Ω. Suppose on the contrary that there are two distinct quadrilateral cells $Q_{u,v}$ and $Q_{w,z}$ such that $T(Q_{u,v}) = T(Q_{w,z})$. Let Λ denote the set $\Omega \backslash \{Q_{u,v}, Q_{w,z}\}$. Then $L = T(L) = T(Q_{u,v}) + T(Q_{w,z}) + \sum_{Q \in \Lambda} T(Q) = T\left(Q_{u,v} + \sum_{Q \in \Lambda} Q\right)$. This is a contradiction because $Q_{u,v} + \sum_{Q \in \Lambda} Q \neq L$.

We suppose further that $T(\Omega) \not\subseteq \Omega$. Since the images of the quadrilateral cells are not zero, it follows that there exists a quadrilateral cell $Q_{p,q}$ such that $T(Q_{p,q}) = Q_{u_1,v_1} + \cdots + Q_{u_m,v_m}$ for some $m \geq 2$ distinct quadrilateral cells. Note that $m < |\Omega|$ because T strongly preserves the matrix L. Let Δ denote the set $\Omega \backslash \{Q_{u_1,v_1}, \ldots, Q_{u_m,v_m}\}$. Since $T(L) = L$, for each $Q_{i,j} \in \Delta$, there exists $S_{i,j} \in \Omega$ such that $T(S_{i,j}) \geq Q_{i,j}$. Let Υ be the collection of such fixed $S_{i,j}$'s. Thus $|\gamma| \leq |\Omega| - m$. Then $T\left(Q_{p,q} + \sum_{S \in \Upsilon} S\right) = T(Q_{p,q}) + T\left(\sum_{S \in \Upsilon} S\right) = Q_{u_1,v_1} + \cdots + Q_{u_m,v_m} + T\left(\sum_{S \in \Upsilon} S\right) \geq Q_{u_1,v_1} + \cdots + Q_{u_m,v_m} + \sum_{Q \in \Delta} Q = L$. This contradicts the fact that T strongly preserves L. Hence $T(\Omega) \subseteq \Omega$.

Therefore, it follows by the finiteness of Ω that T is surjective on Ω.

Next, we prove one of the main results of this section.

Theorem 3.3. *Let* $T : BS_n^{(0)}(\mathscr{B}) \to BS_n^{(0)}(\mathscr{B})$ *be a linear transformation. Then*

(i) T *preserves double-star matrices, and*

(ii) T *strongly preserves the matrix* L

if and only if T *is induced by* (σ, F) *where* σ *is a half-permutation and* F *is a tetrasymmetric matrix.*

Proof. Note that T is bijective on Ω because of the properties of σ and F. Hence, the sufficient part follows.

To show the necessary part, we first show that for each nonzero matrix $X \in BS_n^{(0)}(\mathscr{B})$, $T(X)$ is not the zero matrix. Suppose on the contrary that there is a nonzero matrix X such that $T(X)$ is zero.

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Since X is nonzero, there is a quadrilateral cell $Q \leq X$ such that $T(Q)$ is zero. This is a contradiction because Q is a double-star matrix. Hence T is bijective on Ω by Lemma3.2.

First, we consider the case that $n \leq 5$. There is nothing to do with the case $n = 1, 2$ because $BS_1^{(0)}$ and $BS_2^{(0)}$ are the sets of the zero matrix.

For $n = 3$, we have $\Omega_3 = \{Q_{1,2}\}$. Since T is bijective on Ω_3, it follows that $T(Q_{1,2}) = Q_{1,2}$, i.e., T is induced by (σ, F) where σ is the identity map on $\mathbb{N}_3$ and F is zero.

For $n = 4$, we have $\Omega_4 = \{Q_{1,2}, Q_{1,3}\}$. This implies that T is the identity map on Ω_4 or $T(Q_{1,2}) = Q_{1,3}$ and $T(Q_{1,3}) = Q_{1,2}$. Consequently, T is induced by (σ, F) where σ is the identity map on $\mathbb{N}_4$ and F is zero or $F = L_4$.

For $n = 5$, we have $\Omega_5 = \{Q_{1,2}, Q_{1,3}, Q_{1,4}, Q_{2,3}\}$. Since T preserves double-star matrices, and each of the three quadrilateral cells summing to DS_1 is mapped to three distinct quadrilateral cells, it follows that $T(DS_1) = DS_1$ or $T(DS_1) = DS_2$. Similarly, $T(DS_2) = DS_1$ or $T(DS_2) = DS_2$ and we also obtain that $T(DS_3) = DS_3$. This implies that T is induced by (σ, F) where σ is the identity map on $\mathbb{N}_5$ or $\sigma = (12)(3)(45)$ and F is zero or $F = Q_{1,2} + Q_{1,4}$.

Now, we consider the case that $n \geq 6$. We next define a permutation σ on $\mathbb{N}_{\lceil \frac{n}{2} \rceil}$. Since T preserves double-star matrices, $T(DS_i)$ is a double-star matrix for all $1 \leq i \leq \lceil \frac{n}{2} \rceil$. That means for each $i \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$, $T(DS_i)$ is dominated by DS_j for some $j \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$. Then we define $\sigma : \mathbb{N}_{\lceil \frac{n}{2} \rceil} \to \mathbb{N}_{\lceil \frac{n}{2} \rceil}$ by $\sigma(i) = j$, if $T(DS_i) \leq DS_j$ for all $i \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$.

To show that σ is well-defined, we suppose that there exist $i \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$ such that $T(DS_i) \leq DS_p$ and $T(DS_i) \leq DS_q$ for some $p, q \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$ with $p \neq q$. Then, it follows from Lemma 3.1(ii) that $T(DS_i) \leq Q_{p,q} + Q_{p,n-q+1}$. Next, we calculate the numbers of quadrilateral cells that are summed to L and $L \backslash DS_i$. Note that

$$|\Omega_L| = \begin{cases} \frac{(n-1)^2}{4}, & \text{if } n \text{ is odd;} \\ \frac{n(n-2)}{4}, & \text{if } n \text{ is even.} \end{cases}$$

We observe that if n is odd and , then $|\Omega_{DS_i}| = \frac{n-1}{2}$, otherwise $|\Omega_{DS_i}| = n - 2$. Since $n \geq 6$, in both cases, the number of all quadrilateral cells that are summed to $L \backslash DS_i$ is less than the number of quadrilateral cells excluding $Q_{p,q}$ and $Q_{p,n-q+1}$. Since $L = T(L) = T(L \backslash DS_i) + T(DS_i) \leq T(L \backslash DS_i) + Q_{p,q} + Q_{p,n-q+1}$, by the simple pigeonhole principle, the image of some quadrilateral cells dominates at least two quadrilateral cells. Then, there exists a quadrilateral cell $Q_{x,y}$ such that $T(Q_{x,y}) \geq Q_{u,v} + Q_{w,z}$ where $u \leq v$, $w \leq z$, $u, w \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$ and $(u, v) \neq (w, z)$. Since $T(Q_{x,y})$ is a double-star matrix, there exists $k \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$ such that $T(Q_{x,y}) \leq DS_k$. Hence, $Q_{u,v} \leq Q_{u,v} + Q_{w,z} \leq T(Q_{x,y}) \leq DS_k$. Then, $k = u$ or either $k = v$ if $v \leq \lceil \frac{n}{2} \rceil$, or $k = n - v + 1$ if $v > \lceil \frac{n}{2} \rceil$. For convenience, we may assume that $v \leq \lceil \frac{n}{2} \rceil$. We then separate our proof into two cases and two subcases therein.

Case 1. $DS_u \geq Q_{u,v} + Q_{w,z}$.

Subcase 1.1. $DS_v Q_{u,v} + Q_{w,z}$.

This means that DS_u is the only double-star matrix that dominates $Q_{u,v} + Q_{w,z}$. Since $Q_{u,v} + Q_{w,z} \leq T(Q_{x,y}) \leq T(DS_x)$ and $T(DS_x)$ is a double-star matrix, it follows that $T(DS_x) \leq DS_u$. Similarly, $T(DS_y) \leq DS_u$. Note that we can consider DS_{n-y+1} instead of DS_y if $y > \lceil \frac{n}{2} \rceil$. Notice that the matrix L can be written as $L = DS_x + DS_y + DS_{z_1} + \cdots + DS_{z_{\lceil \frac{n}{2} \rceil - 3}}$ for some $z_1, \ldots, z_{\lceil \frac{n}{2} \rceil - 3} \in \mathbb{N}_{\lceil \frac{n}{2} \rceil} \backslash \{x, y\}$. Then

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$$L = T(L) = T(DS_x) + T(DS_y) + T(DS_{z_1}) + \cdots + T\left(DS_{z_{\lceil \frac{n}{2} \rceil - 3}}\right)$$

$$\leq DS_u + T(DS_{z_1}) + \cdots + T\left(DS_{z_{\lceil \frac{n}{2} \rceil - 3}}\right).$$

Thus, L is dominated by the sum of at most $\lceil \frac{n}{2} \rceil - 2$ full-star matrices. This is a contradiction.

Subcase 1.2. $DS_v \geq Q_{u,v} + Q_{w,z}$.

Then, $Q_{u,v} + Q_{w,z} \leq DS_u + DS_v$. Since $Q_{u,v} + Q_{u,n-v+1}$ is the only sum of two distinct quadrilateral cells dominated by $DS_u + DS_v$, we obtain that $Q_{w,z} = Q_{u,n-v+1}$. Thus, $Q_{u,v} + Q_{u,n-v+1} \leq T(Q_{w,z}) \leq T(DS_x)$. It follows that $T(DS_x) \leq DS_u$ or $T(DS_x) \leq DS_v$. Similarly, $T(DS_y) \leq DS_u$ or $T(DS_y) \leq DS_v$.

If $T(DS_x) \leq DS_u$ and $T(DS_y) \leq DS_u$, then we get a contradiction as the subcase 1.1. Similarly, $T(DS_x) \leq DS_u$ and $T(DS_y) \leq DS_u$ cannot occur simultaneously. This leaves us either (i) $T(DS_x) \leq DS_u$ and $T(DS_y) \leq DS_v$ or (ii) $T(DS_x) \leq DS_v$ and $T(DS_y) \leq DS_u$. In any cases, $T(Q_{x,y}) \leq Q_{u,v} + Q_{u,n-v+1}$ because $Q_{x,y}$ is dominated by both DS_x and DS_y and T is linear. Hence, $Q_{u,v} + Q_{u,n-v+1} = Q_{u,v} + Q_{w,z} \leq T(Q_{x,y}) \leq Q_{u,v} + Q_{u,n-v+1}$. Thus, $T(Q_{x,y}) = Q_{u,v} + Q_{u,n-v+1}$. This is a contradiction since T is injective on Ω_n.

Case 2. $DS_u Q_{u,v} + Q_{w,z}$.

Subcase 2.1. $DS_v Q_{u,v} + Q_{w,z}$. This case is impossible.

Subcase 2.2. $DS_v \geq Q_{u,v} + Q_{w,z}$. In this case, we get a contradiction similarly to the subcase 1.1.

Now we can conclude that σ is well-defined.

By Lemma, we have that T is bijective on Ω and since T is linear, T is bijective on $BS_n^{(0)}$. This means that T maps $\{DS_1, \ldots, DS_{\lceil \frac{n}{2} \rceil}\}$ onto $\{DS_1, \ldots, DS_{\lceil \frac{n}{2} \rceil}\}$ injectively, and $T(DS_i) = DS_{\sigma(i)}$. Indeed, if n is odd, then $T(DS_{\lceil \frac{n}{2} \rceil}) = DS_{\lceil \frac{n}{2} \rceil}$. Hence, σ is a permutation on $\mathbb{N}_{\lceil \frac{n}{2} \rceil}$. Then, we extend σ to be a half-permutation on $\mathbb{N}_n$. That is, $\sigma(i) = n - \sigma(n - i + 1) + 1$ for all $\lceil \frac{n}{3} \rceil \leq i \leq n$.

Let consider the image of the quadrilateral cells. Let $Q_{i,j} \in \Omega$. Without loss of generality, we assume that $i,j \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$. Then, $T(Q_{i,j}) \leq T(DS_i) = DS_{\sigma(i)}$ and $T(Q_{i,j}) \leq T(DS_j) = DS_{\sigma(j)}$. By Lemma 3.1(ii), we obtain that $T(Q_{i,j}) \leq Q_{\sigma(i),\sigma(j)} + Q_{\sigma(i),n-\sigma(j)+1}$. Since T sends injectively quadrilateral cells to quadrilateral cells, either $T(Q_{i,j}) = Q_{\sigma(i),\sigma(j)}$ or $T(Q_{i,j}) = Q_{\sigma(i),n-\sigma(j)+1}$.

If $T(Q_{i,j}) = Q_{\sigma(i),\sigma(j)}$, then we can conclude that $T(Q_{p,q}) = Q_{\sigma(p),\sigma(q)}$ for all $(p,q) \in \{(i,j),(j,i),(n-j+1,n-i+1),(n-i+1,n-j+1)\}$ and $T(Q_{p,q}) = Q_{\sigma(p),\sigma(q)}$ for all $(p,q) \in \{(i,n-j+1),(n-j+1,i),\ (j,n-i+1),(n-i+1,j)\}$. Moreover, we obtain the similar result for the case that $T(Q_{i,j}) = Q_{i,n-\sigma(j)+1}$. Hence, for each $i,j \in \mathbb{N}_n$ with $i \neq j$, it follows that either

(i) if $T(Q_{i,j}) = Q_{\sigma(i),\sigma(j)}$, then $T(Q_{p,q}) = Q_{\sigma(p),\sigma(q)}$ for all $(p,q) \in \Lambda_{i,j}$, or

(ii) if $T(Q_{i,j}) = Q_{\sigma(i),n-\sigma(j)+1}$, then $T(Q_{p,q}) = Q_{\sigma(p),n-\sigma(q)+1}$ for all $(p,q) \in \Lambda_{i,j}$.

We now construct an $n \times n$ Boolean matrix $F = [f_{i,j}]$ by letting

(i) $f_{i,i} = 0 = f_{i,n-i+1}$ for all $i \in \mathbb{N}_n$;

(ii) if n is odd, $f_{i,j} = 0$ whenever $i = \lceil \frac{n}{2} \rceil$ or $j = \lceil \frac{n}{2} \rceil$;

(iii) $f_{ij} = \begin{cases} 0, & \text{if } T(Q_{ij}) = Q_{\sigma(i),\sigma(j)} \\ 1, & \text{if } T(Q_{ij}) = Q_{\sigma(i),n-\sigma(j)+1} \end{cases}$ for all $i,j \in \mathbb{N}_n$ and $i \neq j$.

Then, F is tetrasymmetric. Let $Q_{ij} \in \Omega$. Then

(i) $T(Q_{ij}) = Q_{\sigma(i),\sigma(j)}$ implies $T(Q_{ij}) = P_\sigma Q_{ij} (Q_\sigma^{(0)})^t$; and

(ii) $T(Q_{ij}) = Q_{\sigma(i),n-\sigma(j)+1}$ implies $T(Q_{ij}) = P_\sigma Q_{ij} (P_\sigma^{(1)})^t$.

Thus, for each $A \in BS_n^{(0)}$, $T(A) = \sum_{Q_{ij} \in \Omega_A} P_\sigma Q_{ij} (P_\sigma^{(f_{ij})})^t$. That is T is induced by (σ, F), where σ is a half-permutation and F is a tetrasymmetric matrix.

From Lemma 3.1(i), we can conclude that T preserves the set of all matrices of term ranks 2 and 4 if and only if T preserves double-star matrices. We observe that if T is induced by (σ, F), where σ is a half-permutation and F is a tetrasymmetric matrix, then T preserves term ranks 2 and 4. By these facts and the previous theorem, we obtain immediately the following results. We also provide the number of linear transformations satisfying such assumptions.

Corollary 3.4 *Let $T : BS_n^{(0)}(\mathscr{A}) \to BS_n^{(0)}(\mathscr{B})$ be a linear transformation. Then*

(i) T preserves term ranks 2 and 4, and

(ii) T strongly preserves the matrix L

if and only if T is induced by (σ, F) where σ is a half-permutation and F is a tetrasymmetric matrix.

Moreover, the number of such linear transformations is $\lfloor \frac{n}{2} \rfloor ! \frac{1}{2} \left(\lfloor \frac{n}{2} \rfloor^2 - \lfloor \frac{n}{2} \rfloor \right)$.

Next, we provide the characterization of double-star cover number preservers. Note that when $n \leq 4$, we obtain the characterization of linear transformations on $BS_n^{(0)}(\mathscr{A})$ preserving double-star cover number easily. From now on, we assume that $n \geq 5$.

Theorem 3.5. *Let $T : BS_n^{(0)}(\mathscr{A}) \to BS_n^{(0)}(\mathscr{B})$ be a linear transformation. Then, T preserves double-star cover numbers 1 and 2 if and only if T is induced by (σ, F, G) where σ is a half-mapping on $\mathbb{N}_n$, $F, G \in BS_n^{(0)}(\mathscr{A})$ with $F \leq G$ and $F <_s G$ provided n is odd.*

Moreover, the number of such linear transformations is

$$\begin{cases} \frac{n}{2}! 3^{\frac{n(n-2)}{4}}, & \text{if n is even,} \\ \lfloor \frac{n}{2} \rfloor ! \left(3^{\frac{(n-1)(n-3)}{4}} + \lfloor \frac{n}{2} \rfloor 3^{\frac{(n-3)^2}{4}} \right), & \text{if n is odd.} \end{cases}$$

Proof . The sufficient part is obvious. To show the necessary part, we first define a permutation on $\mathbb{N}_{\lceil \frac{n}{2} \rceil}$. Since T preserves double-star cover number 1, for each $i \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$, there exists $j \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$ such that $T(DS_i) \leq DS_j$. We then define $\sigma : \mathbb{N}_{\lceil \frac{n}{2} \rceil} \to \mathbb{N}_{\lceil \frac{n}{2} \rceil}$ by $\sigma(i) = j$ if $T(DS_i) \leq DS_j$ for all $i \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$.

To show that σ is well-defined, suppose that there exists $i \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$ such that $T(DS_i) \leq DS_p$ and $T(DS_i) \leq DS_q$ where $p, q \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$ with $p \neq q$. It follows from Lemma 3.1(ii) that $T(DS_i) Q_{p,q} + Q_{p,n-q+1}$. Let $j, k \in \mathbb{N}_{\lceil \frac{n}{2} \rceil} \setminus \{i\}$ with $j \neq k$ and $|j - i| = 1$. Then $T(Q_{ij} + Q_{j,k}) = T(Q_{ij}) + T(Q_{j,k}) \leq T(DS_i) + T(Q_{j,k}) \leq Q_{p,q} + Q_{p,n-q+1} + T(Q_{j,k})$. Since $Q_{i,j} + Q_{j,k}$ is of double-star cover number 1, it follows that $T(Q_{j,k}) \leq DS_p$ or $T(Q_{j,k}) \leq DS_q$. We consider only the case that $T(Q_{j,k}) \leq DS_p$ because the other case is obtained similarly. Then $T(DS_i + Q_{j,k}) = T(DS_i) + T(Q_{j,k}) \leq Q_{p,q} + Q_{p,n-q+1} + DS_p$. This is a contradiction

because $DS_i + Q_{j,k}$ is of double-star cover number 2, but $Q_{p,q} + Q_{p,n-q+1} + DS_p$ is of double-star cover number 1. Hence σ is well-defined on $\mathbb{N}_{\lceil \frac{n}{2} \rceil}$. Since T preserves double-star cover number 2, it implies that σ is a permutation on $\mathbb{N}_{\lceil \frac{n}{2} \rceil}$. Next, we extend σ to be a half-mapping on $\mathbb{N}_n$.

Now we consider the images of the quadrilateral cells. Let $Q_{i,j} \in \Omega$. Without loss of generality, we may assume that $i,j \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$. Then, $T(Q_{i,j}) \leq T(DS_i) \leq DS_{\sigma(i)}$ and $T(Q_{i,j}) \leq T(DS_j) \leq DS_{\sigma(j)}$. This implies that $T(Q_{i,j}) \leq Q_{\sigma(i),\sigma(j)} + Q_{\sigma(i),n-\sigma(j)+1}$. Since $T(Q_{i,j})$ is nonzero, Boolean matrices $F = [f_{i,j}]$ and $G = [g_{i,j}]$ can be constructed as follows. Let $f_{i,i} = 0 = f_{i,n-i+1}$ for all $i \in \mathbb{N}_n$ and

$$f_{i,j} = \begin{cases} 0, & \text{if } T(Q_{i,j}) \geq Q_{\sigma(i),\sigma(j)}, \\ 1, & \text{if } T(Q_{i,j}) Q_{\sigma(i),\sigma(j)} \end{cases}$$

for all $i,j \in \mathbb{N}_n$ with $i \neq j$. Also, let $g_{i,i} = 0 = g_{i,n-i+1}$ for all $i \in \mathbb{N}_n$ and

$$g_{i,j} = \begin{cases} 0, & \text{if } T(Q_{i,j}) Q_{\sigma(i),n-\sigma(j)+1}, \\ 1, & \text{if } T(Q_{i,j}) \geq Q_{\sigma(i),n-\sigma(j)+1} \end{cases}$$

for all $i,j \in \mathbb{N}_n$ with $i \neq j$. Note that if $f_{i,j} = 1$, then $g_{i,j} = 1$ because $T(Q_{i,j}) = Q_{\sigma(i),n-\sigma(j)+1}$. This implies that $F \leq G$. We can conclude that

(i) if $T(Q_{i,j}) = Q_{\sigma(i),\sigma(j)}$, then $T(Q_{i,j}) = P_\sigma Q_{i,j} (P_\sigma^{(0)} + P_\sigma^{(0)})^t$; and

(ii) if $T(Q_{i,j}) = Q_{\sigma(i),n-\sigma(j)+1}$, then $T(Q_{i,j}) = P_\sigma Q_{i,j} (P_\sigma^{(1)} + P_\sigma^{(1)})^t$; and

(iii) if $T(Q_{i,j}) = Q_{\sigma(i),\sigma(j)} + Q_{\sigma(i),n-\sigma(j)+1}$, then $T(Q_{i,j}) = P_\sigma Q_{i,j} (P_\sigma^{(0)} + P_\sigma^{(1)})^t$.

Thus, for each $A \in BS_n^{(0)}$, it follows that

$$T(A) = \sum_{Q_{ij} \in \Omega_A} P_\sigma Q_{ij} (P_\sigma^{(f_{ij})} + P_\sigma^{(g_{ij})})^t.$$

Consequently, T is induced by (σ, F, G), where σ is a half-mapping on $\mathbb{N}_n$ and $F, G \in BS_n^{(0)}(\mathscr{B})$ with $F \leq G$.

Furthermore, if n is odd, then there exists $s \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$ such that $\sigma(s) = \lceil \frac{n}{2} \rceil$. It follows that, for each $t \neq s$, $Q_{\sigma(s),\sigma(t)} = Q_{\sigma(s),n-\sigma(t)+1}$. That is $f_{s,t} = 0$ and $g_{s,t} = 1$ for all $t \neq s$. Hence $F <_s G$.

Next, we count the number of such linear transformations. We say that $\Lambda_{i,j}$ is *free* if $(i,j) \notin \bigcup_{t \neq s} \Lambda_{s,t}$. The following table shows the possible elements of $F_{\Lambda_{i,j}}$ and $G_{\Lambda_{i,j}}$ when $\Lambda_{i,j}$ is free.

F		G	
$f_{i,j}$	$f_{i,n-j+1}$	$g_{i,j}$	$g_{i,n-j+1}$
0	0	0,1	0,1
0	1	0,1	1
1	0	1	0,1
1	1	1	1

Case 1. n is even. Since the number of half-mappings σ is $\frac{n}{2}!$ and there are $\frac{n(n-2)}{8}$ sets of $\Lambda_{i,j}$, the number of such linear transformations is $\frac{n}{2}! 3^{\frac{n(n-2)}{4}}$.

Case 2. n is odd. If $\sigma(\lceil \frac{n}{2} \rceil) = \lceil \frac{n}{2} \rceil$, then the number of half-mappings σ is $\lfloor \frac{n}{2} \rfloor!$ and there are $\frac{(n-1)(n-3)}{8}$ sets of $\Lambda_{i,j}$ which is free. Hence, there exist $\lfloor \frac{n}{2} \rfloor! 3^{\frac{(n-1)(n-3)}{4}}$ such possible linear transformations in this case. On the other hand, if $\sigma(\lceil \frac{n}{2} \rceil) \neq \lceil \frac{n}{2} \rceil$, then the number of half-mappings σ is $\lceil \frac{n}{2} \rceil! - \lfloor \frac{n}{2} \rfloor!$ and the number of $\Lambda_{i,j}$ being free is $\frac{n-3}{2}$ whenever $i = \lceil \frac{n}{2} \rceil$ and $\frac{(n-3)(n-5)}{8}$, otherwise. This implies

that the number of such possible linear transformations in this case is $\lfloor \frac{n}{2} \rfloor ! \lfloor \frac{n}{2} \rfloor 3^{\frac{(n-3)^2}{4}}$. Therefore, if n is odd, there are $\lfloor \frac{n}{2} \rfloor ! \left(3^{\frac{(n-1)(n-3)}{4}} + \lfloor \frac{n}{2} \rfloor 3^{\frac{(n-3)^2}{4}} \right)$ such linear transformations.

We investigate further that if the condition `the linear transformation T preserves the matrix L' is assumed, then, in the proof of Theorem 3.5, the entries of matrices F and G are obtained as follows. For each $i,j \in \mathbb{N}_{\lceil \frac{n}{2} \rceil}$, if $1 \in F_{\Lambda_{ij}}$, then $G_{\Lambda_{ij}} = \{1\}$ and $G_{\Lambda_{ij}} \subseteq \{0,1\}$, otherwise. This leads us to the following corollary.

Corollary 3.6. *Let $T : BS_n^{(0)}(\mathscr{B}) \to BS_n^{(0)}(\mathscr{B})$ be a linear transformation. Then*

(i) T preserves double-star cover numbers 1 and 2

(ii) $T(L) = L$

if and only if T is induced by (σ, F, G) where σ is a half-mapping on $\mathbb{N}_n$, $F \leq_\Lambda G$ and $F <_s G$ provided n is odd.

Moreover, the number of such linear transformations is

$$\begin{cases} \lfloor \frac{n}{2} \rfloor ! \left(7^{\frac{(n-1)(n-3)}{8}} 7^{\frac{n(n-2)}{8}} + \lfloor \frac{n}{2} \rfloor 3^{\lfloor \frac{n}{2} \rfloor - 1} 7^{\frac{(n-3)(n-5)}{8}} \right) & \text{if } n \text{ is even,} \\ \text{if } n \text{ is odd.} \end{cases}$$

The following lemma shows the relation between the term rank preservers and the double-star cover number preservers. The proof is done by using Lemma 3.1(iii) and considering the image of each quadrilateral cell.

Lemma 3.7. *Let $T : BS_n^{(0)}(\mathscr{B}) \to BS_n^{(0)}(\mathscr{B})$ be a linear transformation. If T preserves double-star matrices and $T(L) = L$, then T preserves double-star cover numbers 1 and 2.*

The converse of Lemma 3.7 does not hold as the following example shows.

Example 3.8. Let $\sigma : \mathbb{N}_7 \to \mathbb{N}_7$ be defined by $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}$. Moreover, let $F = Q_{1,3}$ and $G = L \backslash Q_{1,5}$. Consider the linear transformation T on $BS_7^{(0)}(\mathscr{B})$ induced by (σ, F, G).

Then $T(Q_{1,2}) = Q_{2,4}$, $T(Q_{1,3}) = Q_{2,3}$, $T(Q_{1,4}) = Q_{1,2} + Q_{1,6}$, $T(Q_{1,5}) = Q_{2,3}$, $T(Q_{1,6}) = Q_{2,4}$, $T(Q_{2,3}) = Q_{3,4}$, $T(Q_{2,4}) = Q_{1,4}$, $T(Q_{2,5}) = Q_{2,4}$, $T(Q_{3,4}) = Q_{1,3} + Q_{1,5}$ and $T(L) = L \backslash Q_{2,5}$.

The next corollary is obtained from Lemma 3.1(i), Corollary 3.6 and Lemma 3.7.

Corollary 3.9. *Let $T : BS_n^{(0)}(\mathscr{B}) \to BS_n^{(0)}(\mathscr{B})$ be a linear transformation. Then, the following are equivalent:*

(i) T preserves double-star matrices and $T(L) = L$;

(ii) T preserves the set of all matrices of term ranks 2 and 4, and $T(L) = L$;

(iii) T preserves double-star cover numbers 1 and 2, and $T(L) = L$;

(iv) T is induced by (σ, F, G) where σ is a half-mapping on $\mathbb{N}_n$, $F \leq_\Lambda G$ and $F <_s G$ provided n is odd.

4. Term rank preservers of bisymmetric matrices over antinegative semirings

Let $A = [a_{ij}] \in BS_n^{(0)}(\mathscr{S})$. Beasley and Pullman (Beasley & Pullman, 1987) defined the *pattern* $\bar{A}$ of A to be the Boolean matrix whose (i,j)th entry is 0 if and only if $a_{ij} = 0$. We say that A has an *L-pattern* whenever $\bar{A} = L$. Note that the term ranks of A and $\bar{A}$ are equal and also the star cover numbers of A and $\bar{A}$. For a linear transformation T on $BS_n^{(0)}(\mathscr{S})$, define $\bar{T} : BS_n^{(0)}(\mathscr{B}) \to BS_n^{(0)}(\mathscr{B})$ by $\bar{T}(\bar{A}) = \sum_{Q_{ij} \in \Omega_{\bar{A}}} \overline{T(Q_{ij})}$ for allA BS_n^(0)($\mathscr{S}$).

The following lemma is used to extend the results in $BS_n^{(0)}(\mathscr{B})$ to $BS_n^{(0)}(\mathscr{S})$. Note that the proof of this lemma is straightforward so that it is left to the reader.

Definition 4.1. Let $T : BS_n^{(0)}(\mathscr{S}) \to BS_n^{(0)}(\mathscr{S})$ be a linear transformation. We say that T *is induced by* (σ, B, F, G) if there exist a mapping σ on $\mathbb{N}_n$, a matrix $B \in M_n(\mathscr{S})$ and matrices $F = [f_{ij}], G = [g_{ij}] \in M_n(\mathscr{B})$ such that
$T(A) = \sum_{Q_{ij} \in \Omega_{\bar{A}}} P_\sigma(A \circ Q_{ij} \circ B)\left(P_\sigma^{(f_{ij})} + P_\sigma^{(g_{ij})}\right)^t$

for all $A \in BS_n^{(0)}(\mathscr{S})$ where $\Omega_{\bar{A}}$ is the set of all quadrilateral cells summing to $\bar{A}$.
We say that T is *induced by* (σ, B, F) if T is induced by (σ, B, F, G) and $F = G$.

Lemma 4.2. *Let $T : BS_n^{(0)}(\mathscr{S}) \to BS_n^{(0)}(\mathscr{S})$ be a linear transformation. Then, T and $\bar{T}$ preserve the same term rank and the same double-star cover number.*

The following characterizations are obtained by applying the results of the previous section and Lemma 4.2. Simply use the same methods of the proof of Corollary 3.3 in (Beasley et al., 2012) but with quadrilateral cells in place of digon cells.

Proposition 4.3. *Let $T : BS_n^{(0)}(\mathscr{S}) \to BS_n^{(0)}(\mathscr{S})$ be a linear transformation. Then the following are equivalent:*

(i) T preserves double-star matrices and T strongly preserves L-pattern;

(ii) T preserves term ranks 2 and 4, and T strongly preserves L-pattern;

(iii) T is induced by (σ, B, F) where σ is a half-permutation, F is a tetrasymmetric matrix and $B \in BS_n^{(0)}(S)$ is the matrix of L-pattern.

Proposition 4.4. *Let $T : BS_n^{(0)}(\mathscr{S}) \to BS_n^{(0)}(\mathscr{S})$ be a linear transformation. Then the following are equivalent:*

(i) T preserves double-star matrices and T preserves L-pattern;

(ii) T preserves the set of all matrices of term ranks 2 and 4, and T preserves L-pattern;

(iii) T preserves double-star cover numbers 1 and 2, and T preserves L-pattern;

(iv) T is induced by (σ, B, F, G) where σ is a half-mapping on $\mathbb{N}_n$, $F \leq_\Lambda G$ and $F <_s G$ provided n is odd and $B \in BS_n^{(0)}(\mathscr{S})$ is the matrix of L-pattern.

5. Conclusion

A new standard form is carefully given in order to characterize linear transformations preserving term ranks of bisymmetric matrices over semirings with some certain conditions. In this research, we investigate term rank preservers on bisymmetric Boolean matrices. The number of such linear transformations is also determined. Besides, the results on Boolean matrices are extended to matrices over semirings. Moreover, the characterizations of linear transformations preserving the special kind of the term rank, which is called the double-star cover number, on bisymmetric matrices over semirings are provided.

In our opinion, many questions can be further studied as shown in the following examples.

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(1) Is it possible to drop the condition that all entries of diagonal and antidiagonal lines of bisymmetric matrices must be zero?

(2) Are there other characterizations of term rank preservers on bisymmetric matrices?

(3) What are characterizations of other rank preservers on bisymmetric matrices?

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