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On Tensor Products of Boolean Semi-modules and Their Term Rank Preservers

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Abstract. We introduce the notion of term rank of an element in tensor products of Boolean semi-modules. We also characterize the linear transformation which generalizes the classical term ranks preserving the Boolean matrices.

Keywords: Linear preserver problem; Boolean matrix; Semi-module; Term rank.

1. Introduction

Linear preserver problems (LPPs) study when linear transformations on spaces of matrices leave certain conditions invariant. While LPPs of many properties of matrices over various algebraic structures have been explored (see [7]), rank one preserver problems play a pivotal role in investigating questions regarding other preservers.

LPPs over Boolean matrices whose entries are in the Boolean semiring $\mathscr{B}$, the set of two elements 0 and 1 with the same arithmetic as for natural numbers

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except that $1+1=1$, was begun in 1980's by Beasley and Pullman [1]. They first provided characterizations of linear transformations preserving the factor ranks one and two. Note that the factor rank of an $m \times n$ Boolean matrix A is the least integer k such that there exist an $m \times k$ and a $k \times n$ Boolean matrices B and C with $A = BC$. Afterward, factor rank preservers of Boolean matrices were generalized, see [3] for example. It is known that this definition is one of many equivalent forms of the usual rank of matrices over fields. However, it turns out that there are many nonequivalent definitions of rank functions for Boolean matrices, see [10]. Also, several rank preservers of Boolean matrices have been established, see [4], [6] and [9] for examples. Among many essential different definitions of rank functions of Boolean matrices, the combinatorial approach leads to the *term rank*, which is the minimum number of lines (rows or columns) required to contain all the nonzero entries of a matrix. In 1987, Beasley and Pullman [2] gave a characterization of term rank preservers as follows. For a linear transformation T of $M_{m,n}(\mathscr{B})$, the set of all $m \times n$ Boolean matrices, T preserves term ranks 1 and 2 if and only if T is a (P,Q)-operator, i.e., there exist permutation matrices P and Q such that $T(A) = PAQ$ for all $A \in M_{m,n}(\mathscr{B})$, or $m = n$ and $T(A) = PA^tQ$ for all $A \in M_{m,n}(\mathscr{B})$. Furthermore, Song and Beasley [8] generalized the result for the case that T preserves any two distinct term ranks in 2013.

In 2011, Lim and Tan [5] gave the notion of the spaces that are more general than $M_{m,n}(\mathscr{B})$, called *Boolean semi-modules*. They also generalized, in [1], the factor rank preservers of Boolean matrices to factor rank preservers of Boolean semi-modules. In this paper, we consider the term rank perservers of Boolean semi-modules, which is a generalization of the results of Beasley and Pullman in [2]. We also obtain the characterization of term rank one preservers between subspaces of Boolean matrices as a special case of our main result. This means that the domain of a linear transformation in our preserver theorem may not be the whole space.

2. Definitions and Preliminaries

In this section, we first recall some basic definitions and basic results of the tensor products of Boolean semi-modules and term ranks of elements, which were given in [5].

For each nonempty set I, let $\mathscr{B}_I$ denote the set of all functions f from I to $\mathscr{B}$ such that $\{i \in I : f(i) \neq 0\}$ is a finite set. For any $f, g \in \mathscr{B}_I$, the function $f + g : I \to \mathscr{B}$ is defined by $(f+g)(i) = f(i) + g(i)$ for any $i \in I$. A *Boolean semi-module* is a subset of $\mathscr{B}_I$ which is closed under addition and contains the zero function. Let $U, V \subseteq \mathscr{B}_I$ be Boolean semi-modules. Then U is called a *subsemi-module* of V if $U \subseteq V$. The intersection of all subsemi-modules of U containing its nonempty subset S, denoted by $\langle S \rangle$, is a subsemi-module of U called *the subsemi-module spanned by S*. We call a set $S \subseteq U$ *independent* if $S \neq \{0\}$ when $|S| = 1$ or every element $f \in S$ is not the sum of any finite number of elements in $S \setminus \{f\}$ when $|S| \geqslant 2$. A subset E of U is called *a basis of* U if E is independent and $\langle E \rangle = U$. The empty set is regarded as the basis of the zero Boolean semi-module. Each element of a basis of U is called a *cell* and the cardinality of the basis is called the *dimension* of U. It was proved in [5] that every Boolean semi-module has a unique basis. For convenience, we denote the basis of U by $\mathcal{E}_U$.

Let $f, g \in \mathscr{B}_I$. The equivalence relation $f \geq g$ is defined by $f + g = f$. Moreover, f and g are said to be *comparable* if $f \geq g$ or $g \geq f$. A nonempty subset H of U is called *non-dominating* when for any nonempty finite subset S of H, $u \in H \setminus S$ implies $\sum_{v \in S} v \ngeq u$. In [5], the authors also showed that a Boolean semi-module with the non-dominating basis still preserves many significant properties which are analogous to ones in a vector space.

A mapping $T : U \to V$ is called a *linear transformation* if it preserves zero and sums. A linear transformation $T : U \to V$ is said to *preserve the property* P if $T(A)$ is of the property P whenever A is of the property P for all $A \in U$.

For any $f \in \mathscr{B}_I$ and $g \in \mathscr{B}_J$, let $f \otimes g$ be the function from $I \times J$ to $\mathscr{B}$ such that $(f \otimes g)(i, j) = f(i)g(j)$ for any $(i, j) \in I \times J$. Then the map $f \otimes g$ is in $\mathscr{B}_{I \times J}$ and is called a *decomposable element* and if f or g is a cell, then $f \otimes g$ is called a *cell decomposable element*. The subsemi-module of $\mathscr{B}_{I \times J}$ spanned by all decomposable elements $f \otimes g$ with $f \in U$ and $g \in V$ is denoted by $U \otimes V$ and is called the *tensor product* of U and V.

Lemma 2.1. [5] *Let* $f_i \otimes g_i, u_j \otimes v_j \in U \otimes V$ *for all* i *and* j. *If* $\sum_{i=1}^{m} f_i \otimes g_i \geq \sum_{j=1}^{n} u_j \otimes v_j$, *then* $\sum_{i=1}^{m} f_i \geq \sum_{j=1}^{n} u_j$ *and* $\sum_{i=1}^{m} g_i \geq \sum_{j=1}^{n} v_j$.

Let $T : U \otimes V \to W \otimes Z$ be a linear transformation. Then T is said to be *induced by two linear transformations* if

(i) there exist linear transformations $\theta : U \to W$ and $\varphi : V \to Z$ such that $T(u \otimes v) = \theta(u) \otimes \varphi(v)$ for any $u \otimes v \in U \otimes V$ or

(ii) there exist linear transformations $\theta : U \to Z$ and $\varphi : V \to W$ such that $T(u \otimes v) = \varphi(v) \otimes \theta(u)$ for any $u \otimes v \in U \otimes V$.

For the case (i), T is written by $\theta \otimes \varphi$, and $\theta \breve{\otimes} \varphi$ for the other. Note that T is a (P, Q)-operator when we consider $U = M_{m,1}(\mathscr{B}), V = M_{n,1}(\mathscr{B}), W = M_{k,1}(\mathscr{B})$ and $Z = M_{l,1}(\mathscr{B})$. A linear transformation $\theta : U \to W$ is called an *embedding* if θ sends all distinct cells to all distinct cells.

The following lemma is applied when a subspace of $M_{m,n}(\mathscr{B})$ is considered as a special case of the tensor product of Boolean semi-modules. Throughout this paper, we focus only the case when $m \geqslant n$ because the other can be done by considering the transpose of matrices.

Lemma 2.2. [5] *Let* $P \in M_{m,n}(\mathscr{B})$ *and* $\theta : M_{n,1}(\mathscr{B}) \to M_{m,1}(\mathscr{B})$ *be the linear transformation defined by* $\theta(u) = Pu$ *for all* $u \in M_{n,1}(\mathscr{B})$. *Then* θ *is injective if and only if* P *contains an* $n \times n$ *permutation submatrix.*

Let A be a nonzero element in $U \otimes V$ and $k \in \mathbb{N}$. We say that A is of *term rank* k, written as $\tau(A) = k$, if k is the least integer such that

$$A \leq u_1 \otimes v_1 + \cdots + u_k \otimes v_k$$

where $u_i \otimes v_i$ is a cell decomposable element in $U \otimes V$. The term rank of the zero element is defined to be 0.

In fact, if I and J are finite sets of order m and n, respectively, then $\mathscr{B}_I \otimes \mathscr{B}_J$ can be identified naturally with $M_{m,n}(\mathscr{B})$ by considering $u \in \mathscr{B}_I$ and $v \in \mathscr{B}_J$ as $m \times 1$ and $n \times 1$ column matrices, respectively, and define $u \otimes v = u \cdot v^t$. Observe that the definition of term rank clearly coincides with the usual term rank of Boolean matrices.

The next lemma shows that the term rank preserves the direction of two comparable elements.

Lemma 2.3. *Let $A, B \in U \otimes V$. If $A \leq B$, then $\tau(A) \leqslant \tau(B)$.*

Proof. Let $\tau(A+B) = k$. Then $A + B = \sum_{i=1}^{k} u_i \otimes v_i$ where u_i or v_i is a cell. It implies that $A \leq \sum_{i=1}^{k} u_i \otimes v_i$ and then $\tau(A) \leqslant k = \tau(A+B) = \tau(B)$. ∎

Example 2.4. Let $e_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^t, e_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^t$ and $e_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^t$ and U be the Boolean semi-module spanned by $\{e_1, e_2, e_3\}$. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $A \in U \otimes U$ and $A \leq e_3 \otimes e_3$. Hence $\tau(A) = 1$. However, A cannot be written as a decomposable element of which one factor is a cell.

The notion from the previous example that cells e_1 and e_2 can be dominated by e_3 leads us to the following proposition.

Proposition 2.5. *Let U and V have no comparable cells and $A \in U \otimes V$. Then $\tau(A) = 1$ if and only if A is a nonzero cell decomposable element.*

Proof. The sufficiency part is clear. To show the necessity, let $A \leq u \otimes v$ where u or v is a cell. Without loss of generality, we assume that $u \in \mathcal{E}_U$. Then there exist $x_1, \ldots, x_k \in U$ and $y_1, \ldots, y_k \in V$ such that $A = x_1 \otimes y_1 + \cdots + x_k \otimes y_k$. Since $A \leq u \otimes v$, it follows from Lemma 2.1 that $x_1 + \cdots + x_k \leq u$. Note that for each i, there exists $e_i \in \mathcal{E}_U$ such that $e_i \leq x_i \leq u$. Since U has no comparable cells, $e_i = u$ for all i. This implies that $x_i = u$ for any i. Hence $A = u \otimes (y_1 + \cdots + y_k)$. ∎

3. Term Rank Preservers Between Tensor Products of Boolean Semi-modules

In order to present the main theorems of our work, we first prove the following significant result.

Proposition 3.1. *Let A and B be nonzero elements in $U \otimes V$ such that each of A, B and $A + B$ are cell-decomposable elements. If A and B are incomparable, then A and B have a common cell factor.*

Proof. Let $A = u_1 \otimes v_1, B = u_2 \otimes v_2$ and $A + B = u_3 \otimes v_3$ where u_i or v_i is a cell for all i. By Lemma 2.1, it follows that $u_1 + u_2 = u_3$ and $v_1 + v_2 = v_3$. Without loss of generality, we may assume that $u_3 \in \mathcal{E}_U$. It can be implied from the independence of $\mathcal{E}_U$ that $u_3 \leq u_1$ or $u_3 \leq u_2$. Hence $u_2 \leq u_1$ or $u_1 \leq u_2$.

Suppose that $u_2 < u_1$. Since A and B are incomparable, $v_2 \nleq v_1$, so that there exists $j' \in J$ with $v_2(j') = 1$ and $v_1(j') = 0$. Since $A + B = C$, for each $i \in I$, $u_2(i) = u_3(i)v_3(j')$. Then $v_3(j') = 1$ because u_2 is nonzero. Hence $u_2 = u_3$. This is a contradiction since $u_3 = u_2 < u_1 \leq u_1 + u_2 = u_3$. ∎

According to the condition in Proposition 2.5, if we assume further that U and V have no comparable cells, then Proposition 3.1 implies the following result.

Proposition 3.2. *Let U and V have no comparable cells and A and B be two term rank one elements in $U \otimes V$ such that $\tau(A + B) = 1$. If A and B are incomparable, then A and B have a common cell factor.*

From now on, we assume that U, V, W and Z have no comparable cells and all of Boolean semi-modules are of dimensions at least two. For each $e \in \mathcal{E}_U$, $f \in \mathcal{E}_V$, subsemi-modules H and K of U and V, respectively, the subsemi-module $e \otimes K = \{e \otimes k \mid k \in K\}$ is celled a *left cell factor subsemi-module of* $U \otimes V$ and the subsemi-module $H \otimes f = \{h \otimes f \mid h \in H\}$ is called a *right cell factor subsemi-module of* $U \otimes V$.

We now state the following theorem related to the term rank of elements in the linear transformations of Boolean semi-modules.

Theorem 3.3. *Let $T : U \otimes V \to W \otimes Z$ be a linear transformation. If T sends distinct term rank one elements to distinct term rank one elements, then one of the following holds:*

(i) *there exist a cell w of W and a linear transformation $\varphi : U \otimes V \to Z$ such that $T(A) = w \otimes \varphi(A)$ for any $A \in U \otimes V$ where φ is injective on the set of all term rank one elements,*

(ii) *there exist a cell z of Z and a linear transformation $\theta : U \otimes V \to W$ such that $T(A) = \theta(A) \otimes z$ for any $A \in U \otimes V$ where θ is injective on the set of all term rank one elements,*

(iii) *T is induced by two embeddings.*

Proof. We first claim that T sends any left cell factor subsemi-modules of $U \otimes V$ to cell factor subsemi-modules of $W \otimes Z$. Let $e \in \mathcal{E}_U$ and f_1, f_2 be two distinct cells in $\mathcal{E}_V$. Then $T(e \otimes f_1) = w_1 \otimes z_1$ and $T(e \otimes f_2) = w_2 \otimes z_2$ where w_1 or z_1 is a cell and w_2 or z_2 is a cell. Because $\tau(e \otimes (f_1 + f_2)) = 1$, so is $w_1 \otimes z_1 + w_2 \otimes z_2$. Observe that $w_1 \otimes z_1$ and $w_2 \otimes z_2$ are incomparable, indeed, if $w_1 \otimes z_1 \leq w_2 \otimes z_2$, then

$$T(e \otimes f_2) = w_2 \otimes z_2 = w_1 \otimes z_1 + w_2 \otimes z_2 = T(e \otimes (f_1 + f_2))$$

leading to $f_1 \leq f_2$, which is a contradiction. As a result of Proposition 3.2, $w_1 \otimes z_1$ and $w_2 \otimes z_2$ have a common cell factor.

Without loss of generality, we may assume that $w_1 = w_2$ is a cell. Then $z_1 \neq z_2$. Let f be any cell but not f_1 and f_2. Then $T(e \otimes f) = w \otimes z$ where w or z is a cell. Similarly, it can be obtained that $w \otimes z$ and $w_1 \otimes z_1$ have a common cell factor and also $w \otimes z$ and $w_2 \otimes z_2$. That is

(i) either $w = w_1$ or $z = z_1$, and
(ii) either $w = w_2$ or $z = z_2$.

If $w \neq w_1$, then $z = z_1$ so that $w \neq w_2$ leading to $z_1 = z = z_2$, which is a contradiction. Thus $w = w_1 = w_2$. Hence $T(e \otimes V) = w_1 \otimes Z_1$ where

$$Z_1 = \langle \{ z \in Z \mid T(e \otimes f) = w_1 \otimes z \text{ for all } f \in \mathcal{E}_V \} \rangle .$$

By the same previous argument, if $z_1 = z_2$, then $T(e \otimes V) = W_1 \otimes z_1$ for some subsemi-module W_1 of W.

Next, we claim that all left cell factor subsemi-modules of $U \otimes V$ are mapped into either the set of left cell factor subsemi-modules or the set of right cell factor subsemi-modules of $W \otimes Z$. Suppose that there are two distinct cells $e_1, e_2 \in \mathcal{E}_U$ such that $T(e_1 \otimes V) = w \otimes Z_1$ and $T(e_2 \otimes V) = W_1 \otimes z$ where $w \in \mathcal{E}_W$, $z \in \mathcal{E}_Z$, W_1 and Z_1 are subsemi-modules of W and Z, respectively. Then there exists $f \in \mathcal{E}_V$ with $T(e_1 \otimes f) = w \otimes h$ for some $h \in Z_1$ and $h \neq z$. Thus $T(e_2 \otimes f) = g \otimes z$ for some $g \in W$. Since $\tau((e_1 + e_2) \otimes f) = 1$, it can be obtained that $w \otimes h$ and $g \otimes z$ are incomparable and also $w \otimes h$ and $g \otimes z$ have a common cell factor. But $h \neq z$ forcing $w = g$. Similarly, $w \otimes z \in T(e_1 \otimes V)$. Since $w \otimes z \in T(e_1 \otimes V) \cap T(e_2 \otimes V)$, it can be implied that $e_1 = e_2$. This is a contradiction.

Besides, by using the similar argument, T sends all right cell factor subsemi-modules of $U \otimes V$ into either the set of left cell factor subsemi-modules or the set of right cell-factor subsemi-modules of $W \otimes Z$.

Next, we consider only the case that T sends left cell factor subsemi-modules of $U \otimes V$ to left cell factor subsemi-modules of $W \otimes Z$ and the other case can be established similarly.

Case 1. T sends right cell factor subsemi-modules of $U \otimes V$ to left cell factor subsemi-modules of $W \otimes Z$. Let $f \in \mathcal{E}_V$ to be fixed. Then there exist $w_f \in \mathcal{E}_W$ and a subsemi-module K_f of Z such that $T(U \otimes f) = w_f \otimes K_f$. Let $e \in \mathcal{E}_U$. Then $T(e \otimes V) = w_e \otimes Z_e$ for some $w_e \in \mathcal{E}_W$ and a subsemi-module Z_e of Z. Since $e \otimes f \in (e \otimes V) \cap (U \otimes f)$, it follows that $w_e = w_f$. Consequently, $T(e \otimes V) \subseteq w_f \otimes Z$, which implies that $\operatorname{Im}(T) \subseteq w_f \otimes Z$. Thus for any $A \in U \otimes V$, there exists $z_A \in Z$ such that $T(A) = w_f \otimes z_A$. Clearly, z_A is unique.

Define $\varphi : U \otimes V \to Z$ by $\varphi(A) = z_A$ where $T(A) = w_f \otimes z_A$ for any $A \in U \otimes V$. Note that φ is well-defined because z_A is unique and φ is linear by the linearity of T. Therefore, $\varphi : U \otimes V \to Z$ is a linear transformation such that $T(A) = w_f \otimes \varphi(A)$ for any $A \in U \otimes V$. Moreover, φ is injective on the set of all term rank one elements since T sends distinct term rank one elements to distinct term rank one elements.

Case 2. T sends right cell factor subsemi-modules of $U \otimes V$ to right cell factor subsemi-modules of $W \otimes Z$. Let $f \in \mathcal{E}_V$. Then there exist uniquely $z_f \in \mathcal{E}_Z$ and a subsemi-module W_f of W such that $T(U \otimes f) = W_f \otimes z_f$. Let $e \in \mathcal{E}_U$. Then there exist uniquely $w_e \in \mathcal{E}_W$ and a subsemi-module Z_e of Z such that $T(e \otimes V) = w_e \otimes Z_e$. This means that

$$T(e \otimes f) \in (w_e \otimes Z_e) \cap (W_f \otimes z_f).$$

Thus $T(e \otimes f) = w_e \otimes z_f$.

Define a mapping $\theta : \mathcal{E}_U \to W$ by $\theta(e) = w_e$ for all $e \in \mathcal{E}_U$, and a mapping $\varphi : \mathcal{E}_V \to Z$ by $\varphi(f) = z_f$ for all $f \in \mathcal{E}_V$. Clearly, θ and φ are well-defined. Then we extend θ and φ linearly to be linear transformations from U to W and V to Z, respectively.

Let $u \in U$ and $v \in V$. If u and v are nonzero, then $u = e_1 + \cdots + e_m$ where $e_1, \ldots, e_m \in \mathcal{E}_U$, and $v = f_1 + \cdots + f_n$ where $f_1, \ldots, f_n \in \mathcal{E}_V$ so that

$$T(u \otimes v) = \sum_{i,j} T(e_i \otimes f_j) = \sum_{i,j} w_{e_i} \otimes z_{f_j} = \sum_{i,j} \theta(e_i) \otimes \varphi(f_j) = \theta(u) \otimes \varphi(v).$$

If $u = 0$ or $v = 0$, then $T(u \otimes v) = 0 = \theta(u) \otimes \varphi(v)$. Therefore $T = \theta \otimes \varphi$. Moreover, θ and φ are embeddings. ■

Remark 3.4. At the end of the proof of the Case 2, if we assume further that W and Z have non-dominating bases, then it can be obtained that θ and φ are injective embeddings. To show that θ is injective, let $u_1, u_2 \in U$ be such that $\theta(u_1) = \theta(u_2)$. Then $u_1 = e_1 + \cdots + e_m$ and $u_2 = f_1 + \cdots + f_n$ where $e_i, f_j \in \mathcal{E}_U$. Then for each i,

$$\theta(e_i) \leq \sum_{k=1}^{m} \theta(e_k) = \sum_{j=1}^{n} \theta(f_j).$$

Since $\mathcal{E}_W$ is non-dominating, for each i, there exists unique j such that

$$\theta(e_i) = \theta(f_j),$$

i.e., $e_i = f_j$, because θ is an embedding, so that $m \leqslant n$. Similarly, $n \leqslant m$. Thus $u_1 = u_2$.

4. Examples and Main Theorem

In this section, we provide some examples to explain and to illustrate the results given in this paper.

Example 4.1. Let U be a Boolean semi-module with only 2 cells e_1 and e_2 where $e_1 < e_2$. Moreover, let V and W be Boolean semi-modules with non-dominating bases $\{f_1, f_2\}$ and $\{g_1, g_2, g_3\}$, respectively. If $T : U \otimes V \to W \otimes W$ is a linear transformation such that

$$T(e_1 \otimes f_1) = g_1 \otimes g_1, T(e_1 \otimes f_2) = g_1 \otimes g_2,$$

$$T(e_2 \otimes f_1) = g_2 \otimes g_1, T(e_2 \otimes f_2) = g_2 \otimes g_3,$$

then $T(e_1 \otimes (f_1 + f_2)) = g_1 \otimes (g_1 + g_2)$ and $T(e_2 \otimes (f_1 + f_2)) = g_2 \otimes (g_1 + g_3)$. Hence T sends all distinct term rank one elements to distinct term rank one elements. However, the image of T is not a cell factor subsemi-module of $W \otimes W$ and T is not induced by any two embeddings.

The above example shows that the condition 'both U and V have no comparable cells' in Theorem 3.3 is necessary.

Example 4.2. Let $U = M_2(\mathscr{B}), V = M_3(\mathscr{B})$ and Z be the Boolean semi-module spanned by three cells $z_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^t, z_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^t$ and $z_3 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^t$. Let $\theta : U \to U$ be the identity map and $\varphi : V \to Z$ be the linear transformation such that $\varphi(v_j) = z_j$ for all cells $v_j \in \mathcal{E}_V$. Then both θ and φ are embeddings but φ is not injective. Note that the linear transformation $T = \theta \otimes \varphi$ preserves term rank one and sends some distinct term rank one elements to the same term rank one element, e.g.,

$$T\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = T\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

The previous example shows that the converse of Theorem 3.3 is not true. However, it is easy to see that this theorem holds when we assume further that both embeddings in (iii) of Theorem 3.3 are injective. From this observation and Remark 3.4, the following result is established.

Proposition 4.3. *Let $T : U \otimes V \to W \otimes Z$ be a linear transformation where W and Z have non-dominating bases. Then T sends distinct term rank one elements to distinct term rank one elements if and only if one of the following is valid:*

(i) *there exist a cell w of W and a linear transformation $\varphi : U \otimes V \to Z$ such that $T(A) = w \otimes \varphi(A)$, for any $A \in U \otimes V$ where φ is injective on the set of all term rank one elements,*

(ii) *there exist a cell z of Z and a linear transformation $\theta : U \otimes V \to W$ such that $T(A) = \theta(A) \otimes z$, for any $A \in U \otimes V$ where θ is injective on the set of all term rank one elements,*

(iii) *T is induced by two injective embeddings.*

By considering U, V, W and Z in Proposition 4.3 as the spaces of column matrices and applying Lemma 2.2, the following result is obtained.

Corollary 4.4. *Let $T : M_{m,n}(\mathscr{B}) \to M_{k,l}(\mathscr{B})$ be a linear transformation. Then T sends distinct term rank one matrices to distinct term rank one matrices if and only if one of the following is true:*

(i) *there exist $e \in \mathcal{E}_{M_{k,1}(\mathscr{B})}$ and a linear transformation $\varphi : M_{m,n}(\mathscr{B}) \to M_{1,l}(\mathscr{B})$ such that $T(A) = e\varphi(A)$, for all $A \in M_{m,n}(\mathscr{B})$ where φ is injective on the set of all term rank one elements,*

(ii) *there exist $f \in \mathcal{E}_{M_{1,l}(\mathscr{B})}$ and a linear transformation $\theta : M_{m,n}(\mathscr{B}) \to M_{k,1}(\mathscr{B})$ such that $T(A) = \theta(A)f$, for all $A \in M_{m,n}(\mathscr{B})$ where θ is injective on the set of all term rank one elements,*

(iii) *$T(A) = PAQ$ for some $P \in M_{k,m}(\mathscr{B})$ and some $Q \in M_{n,l}(\mathscr{B})$ where P contains an $m \times m$ permutation submatrix and Q contains an $n \times n$ permutation submatrix,*

(iv) *$T(A) = PA^tQ$ for some $P \in M_{k,n}(\mathscr{B})$ and some $Q \in M_{m,l}(\mathscr{B})$ where P contains an $n \times n$ permutation submatrix and Q contains an $m \times m$ permutation submatrix.*

To prove another main result, we need the following lemmas.

Lemma 4.5. *Let W and Z have non-dominating bases. Moreover, assume that $T : U \otimes V \to W \otimes Z$ is a linear transformation preserving term ranks one and two and A and B are two distinct elements of term rank one in $U \otimes V$. If $T(A) = T(B)$, then A and B have no common cell factors.*

Proof. Let $A = u \otimes v$ and $B = x \otimes y$ where u or v is a cell and x or y is a cell. Suppose that $u = x \in \mathcal{E}_U$. Then v and y are incomparable since $A \neq B$. We consider only the case that $v \not\leq y$ and the other case is handled similarly. Then there is $f \in \mathcal{E}_V$ such that $f \leq v$ but $f \not\leq y$. Since T preserves term rank one, $T(u \otimes v) = w \otimes z$ where w or z is a cell.

Suppose that $w \in \mathcal{E}_W$. $T(u \otimes f) = w_0 \otimes z_0$ where w_0 or z_0 is a cell. Since

$$w_0 \otimes z_0 = T(u \otimes f) \leq T(u \otimes v) = w \otimes z,$$

it follows that $w_0 \leq w$ and $z_0 \leq z$. Since $w \in \mathcal{E}_W$ is non-dominating, $w_0 = w$. Let $e \in \mathcal{E}_U \setminus \{u\}$. Then $T(e \otimes f) = w_1 \otimes z_1$ where w_1 or z_1 is a cell. If $w_0 \otimes z_0 \leq w_1 \otimes z_1$, then

$$\begin{aligned} T(u \otimes f + e \otimes f + e \otimes y) &= w_0 \otimes z_0 + w_1 \otimes z_1 + T(e \otimes y) \\ &= w_1 \otimes z_1 + T(e \otimes y) \\ &= T(e \otimes (f + y)), \end{aligned}$$

which is a contradiction since $\tau(u \otimes f + e \otimes f + e \otimes y) = 2$ but $\tau(e \otimes (f + y)) = 1$. If $w_1 \otimes z_1 \leq w_0 \otimes z_0$, then

$$T(e \otimes f + x \otimes y) = w_1 \otimes z_1 + w \otimes z \leq w \otimes z,$$

which is a contradiction because $\tau(e \otimes f + x \otimes y) = 2$. Hence $w_0 \otimes z_0$ and $w_1 \otimes z_1$ are incomparable. It follows from Proposition 3.2 that $w_0 = w_1 \in \mathcal{E}_W$ or $z_0 = z_1 \in \mathcal{E}_Z$. By considering $T(e \otimes f + x \otimes y)$ again, it can be obtained that $z_0 = z_1$ is a cell.

Let $y = \sum_{j=1}^{n} y_j$ for some $y_j \in \mathcal{E}_V$. Then for each j, $T(u \otimes y_j) = \widetilde{w}_j \otimes \widetilde{z}_j$ where $\widetilde{w}_j$ or $\widetilde{z}_j$ is a cell. Since

$$T(u \otimes f) \leq T(u \otimes v) = T(x \otimes y) = T(u \otimes y),$$

it follows that $w_0 \otimes z_0 \leq \widetilde{w}_1 \otimes \widetilde{z}_1 + \cdots + \widetilde{w}_n \otimes \widetilde{z}_n$ so that $z_0 \leq \widetilde{z}_1 + \cdots + \widetilde{z}_n$. Since Z has the non-dominating basis, there exists j' such that

$$T(u \otimes f) = w_0 \otimes z_0 \leq w \otimes \widetilde{z}_{j'} = T(u \otimes y_{j'}).$$

Hence $T(u \otimes f + e \otimes y_{j'}) \leq T((u + e) \otimes y_{j'})$, which is a contradiction since $\tau(u \otimes f + e \otimes y_1) = 2$.

Note that if z is a cell, then the contradiction can be obtained similarly. Therefore, A and B have no common cell-factors. ∎

Finally, we prove the main theorem of this paper.

Theorem 4.6. *Let W and Z have non-dominating bases and $T : U \otimes V \to W \otimes Z$ be a linear transformation. Then T preserves term ranks one and two if and only if T is induced by two embeddings.*

Proof. (Necessity). Claim that T is injective on the set of term rank one elements. Let A and B be two distinct elements of term rank one such that $T(A) = T(B)$. It follows from Lemma 3.9 that A and B have no common cell factors. It can be implied by Proposition 3.2 that $\tau(T(A) + T(B)) = 2$. Then

$$1 = \tau(T(B)) = \tau(T(A) + T(B)) = 2,$$

which is a contradiction. We get our claim. Then Corollary 3.7 and the fact that T preserves term rank two imply that T is induced by two injective embeddings.

(Sufficiency). Without loss of the generality, we assume that $T = \theta \otimes \varphi$. Clearly, T preserves term rank one elements. Let $A \in U \otimes V$ with $\tau(A) = 2$. Then $A \leq u_1 \otimes v_1 + u_2 \otimes v_2$ where $u_i \otimes v_i$ is a cell decomposable element. Hence

$$T(A) \leq T(u_1 \otimes v_1) + T(u_2 \otimes v_2) = \theta(u_1) \otimes \varphi(v_1) + \theta(u_2) \otimes \varphi(v_2).$$

Since θ and φ are embeddings, $0 \neq \tau(T(A)) \leqslant 2$.

Suppose that $\tau(T(A)) = 1$. Then $T(A) = w \otimes z$ where w or z is a cell. Suppose that $w \in \mathcal{E}_W$. Write $A = e_1 \otimes v_1 + \cdots + e_n \otimes v_n$ where $e_i \in \mathcal{E}_U$ and $v_i \in V$. Then

$$T(A) = \theta(e_1) \otimes \varphi(v_1) + \cdots + \theta(e_n) \otimes \varphi(v_n) = w \otimes z.$$

Thus $\theta(e_1) + \cdots + \theta(e_n) = w$. Since $\mathcal{E}_W$ have no comparable cells and θ is an embedding, $e_1 = \cdots = e_n$. Hence $A = e_1 \otimes (v_1 + \cdots + v_n)$, i.e., $\tau(A) = 1$, which is a contradiction. Hence $\tau(T(A)) = 2$. ■

In the proof of the necessary part of Theorem 4.6, we establish further that both two embeddings inducing T are injective. This leads us to obtain the Boolean matrices version of Theorem 4.6. as follows.

Corollary 4.7. *Let $T : M_{m,n}(\mathscr{B}) \to M_{k,l}(\mathscr{B})$ be a linear transformation where* $\min\{m, n, k, l\} \geq 2$. *Then T preserves term ranks one and two if and only if*

(i) *$T(A) = PAQ$ for some $P \in M_{k,m}(\mathscr{B})$ and some $Q \in M_{n,l}(\mathscr{B})$ where P contains an $m \times m$ permutation submatrix and Q contains an $n \times n$ permutation submatrix, or*

(ii) *$T(A) = PA^tQ$ for some $P \in M_{k,n}(\mathscr{B})$ and some $Q \in M_{m,l}(\mathscr{B})$ where P contains an $n \times n$ permutation submatrix and Q contains an $m \times m$ permutation submatrix.*

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