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Some convergence theorems of the Mann iteration for monotone α-nonexpansive mappings

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ABSTRACT

In this paper, we introduce the concept of monotone α-nonexpansive mappings in an ordered Banach space E with the partial order $\leq$, which contains monotone nonexpansive mappings as special case. With the help of the Mann iteration, we show some existence theorems of fixed points of monotone α-nonexpansive mappings in uniformly convex ordered Banach space. Also, we prove some weak and strong convergence theorems of the Mann iteration for finding an order fixed point of monotone α-nonexpansive mappings under the condition

$$\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0 \quad \text{or} \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

1. Introduction

Let T be a mapping with domain $D(T)$ and range $R(T)$ in an ordered Banach space E endowed with the partial order $\leq$. Then T: $D(T) \to R(T)$ is said to be:

(1) *monotone* [1] if $Tx \leq Ty$ for all $x, y \in D(T)$ with $x \leq y$;
(2) *monotone nonexpansive* [1] if T is monotone and

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$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in D(T)$ with $x \le y$.

Clearly, a monotone nonexpansive mapping may be discontinuous.

In 2015, Bachar and Khamsi [1] introduced the concept of a monotone nonexpansive mapping and studied common approximate fixed points of a monotone nonexpansive semigroup. In 2015, Dehaish and Khamsi [2] proved some weak convergence theorems of the Mann iteration for finding some order fixed points of monotone nonexpansive mappings in uniformly convex ordered Banach spaces.

In 1953, Mann [10] introduced the following iteration to finding a fixed point of a nonexpansive mapping T, which is referred to as the *Mann iteration*,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n \tag{1.1}$$

for each $n \ge 1$, where $\{\beta_n\} \subset [0, 1]$ is a sequence satisfying some conditions. In fact, the Mann iteration $\{x_n\}$ converges weakly to a fixed point of a nonexpansive mapping T in a Hilbert space H provided the sequence $\{\beta_n\}$ satisfies the condition

$$\sum_{n=1}^{+\infty} \beta_n(1 - \beta_n) = +\infty.$$

In the past several decades, many mathematical workers have studied the strong and weak convergence of the Mann iteration and its modified version for various types of nonlinear mappings, for instance, George and Nse [5] for demicontractive mappings, Hussain et al. [7] for hemicontractions, Liu [9] for strongly accretive mappings, Narghirad et al. [11] for α-nonexpansive mappings, Suzuki [16] for nonexpansive semigroups, Song [15] for a family of nonexpansive mappings, Opial [13] for nonexpansive mappings, Kim et al. [8] for strictly hemicontractive mappings, Okeke and Kim [12] for random operators, Berinde [3] and George and Shaini [4] for Zamfirescu operators, Gu and Lu [6] for nonlinear variational inclusions problems, Zhou et al. [20] for strictly pseudocontraction mappings, Song and Wang [14], Zhang and Su [19], Zhou [21] for the modified Mann iteration of strictly pseudocontraction mapping and therein.

In this paper, we mainly consider the Mann iteration for monotone α-nonexpansive mappings in an ordered Banach space E with the partial order $\le$. In Section 3, we show some existence theorems of order fixed points in uniformly convex ordered Banach spaces. In Section 4, we prove some weak and strong convergence theorems of the Mann iteration for finding an order fixed point of monotone α-nonexpansive mappings under the condition

$$\limsup_{n\to\infty} \beta_n(1 - \beta_n) > 0 \quad \text{or} \quad \liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0.$$

2. Preliminaries and basic results

Throughout this paper, let E be an ordered Banach space with the norm $\|\cdot\|$ and the partial order $\le$. Let $F(T) = \{x \in E : Tx = x\}$ denote the set of all fixed points of a mapping T: $E \to E$.

Let E be a real Banach space and P be a subset of E. P is called a *closed convex cone* if the following conditions:

(c1) P is nonempty closed and $P \ne \{0\}$;
(c2) for all $a, b \in \mathbb{R}$ with $a, b \ge 0$ and $x, y \in P$, $ax + by \in P$;
(c3) if $x \in P$ and $-x \in P$, then $x = 0$.

Let P be a closed convex cone of a real Banach space E. A *partial order* $\le$ with respect to P in E is defined as follows:

$$x \le y \ (x < y) \ \text{ if and only if } \ y - x \in P \ (y - x \in \mathring{P})$$

for all $x, y \in E$, where $\mathring{P}$ is the interior of P.

In the sequel, we assume that the order intervals are closed and convex. An *order interval* $[x, y]$ for all $x, y \in E$ is given by

$$[x, y] = \{z \in E : x \le z \le y\}. \tag{2.1}$$

Then the convexity of the order interval $[x, y]$ implies that

$$x \le tx + (1 - t)y \le y \tag{2.2}$$

for all $x, y \in E$ with $x \le y$.

Definition 2.1. Let K be a nonempty closed convex subset of an ordered Banach space $(E, \le)$. A mapping T: $K \to E$ is said to be:

(1) *monotone α-nonexpansive* if T is monotone and, for some $\alpha < 1$,

$$\|Tx - Ty\|^2 \le \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2$$

for all $x, y \in K$ with $x \le y$;

(2) *monotone quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \le \|x - p\|$$

for all $p \in F(T)$ and $x \in K$ with $x \le p$ or $x \ge p$.

Obviously, a monotone nonexpansive mapping is monotone 0-nonexpansive.

Lemma 2.2. *Let K be a nonempty closed convex subset of an ordered Banach space $(E, \le)$ and T: $K \to K$ be a monotone α-nonexpansive mapping. Then we have*

(1) *T is monotone quasi-nonexpansive;*
(2) *for all $x, y \in K$ with $x \le y$,*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \frac{2\alpha}{1-\alpha}\|Tx - x\|^2 + \frac{2|\alpha|}{1-\alpha}\|Tx - x\|(\|x - y\| + \|Tx - Ty\|).$$

Proof. (1) It follows from the definition of a monotone α-nonexpansive mapping that

$$\begin{aligned}\|Tx - p\|^2 &= \|Tx - Tp\|^2\\ &\le \alpha\|Tx - p\|^2 + \alpha\|Tp - x\|^2 + (1-2\alpha)\|x - p\|^2\\ &= \alpha\|Tx - p\|^2 + (1-\alpha)\|x - p\|^2\end{aligned}$$

and so

$$\|Tx - p\|^2 \le \|x - p\|^2.$$

(2) If $1 > \alpha \ge 0$, then we have

$$\begin{aligned}\|Tx - Ty\|^2 &\le \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1-2\alpha)\|x - y\|^2\\ &\le \alpha(\|Tx - x\| + \|x - y\|)^2 + \alpha(\|Ty - Tx\| + \|Tx - x\|)^2 + (1-2\alpha)\|x - y\|^2\\ &= \alpha\|Tx - x\|^2 + 2\alpha\|Tx - x\|\|x - y\| + \alpha\|x - y\|^2\\ &\quad + \alpha\|Ty - Tx\|^2 + 2\alpha\|Ty - Tx\|\|Tx - x\| + \alpha\|Tx - x\|^2 + (1-2\alpha)\|x - y\|^2\\ &= 2\alpha\|x - Tx\|^2 + (1-\alpha)\|x_n - x\|^2 + \alpha\|Tx - Tx_n\|^2 + 2\alpha\|Tx - x\|(\|x - y\| + \|Tx - Ty\|)\end{aligned}$$

and so

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \frac{2\alpha}{1-\alpha}\|Tx - x\|^2 + \frac{2\alpha}{1-\alpha}\|Tx - x\|(\|x - y\| + \|Tx - Ty\|).$$

If $\alpha < 0$, then we have

$$\begin{aligned}\|Tx - Ty\|^2 &\le \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1-2\alpha)\|x - y\|^2\\ &\le \alpha(\|Tx - x\| - \|x - y\|)^2 + \alpha(\|Ty - Tx\| - \|Tx - x\|)^2 + (1-2\alpha)\|x - y\|^2\\ &= \alpha\|Tx - x\|^2 - 2\alpha\|Tx - x\|\|x - y\| + \alpha\|x - y\|^2\\ &\quad + \alpha\|Ty - Tx\|^2 - 2\alpha\|Ty - Tx\|\|Tx - x\| + \alpha\|Tx - x\|^2 + (1-2\alpha)\|x - y\|^2\\ &= 2\alpha\|x - Tx\|^2 + (1-\alpha)\|x_n - x\|^2 + \alpha\|Tx - Tx_n\|^2 - 2\alpha\|Tx - x\|(\|x - y\| + \|Tx - Ty\|)\end{aligned}$$

and so

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \frac{2\alpha}{1-\alpha}\|Tx - x\|^2 + \frac{-2\alpha}{1-\alpha}\|Tx - x\|(\|x - y\| + \|Tx - Ty\|).$$

Thus it follows that, for all $x, y \in K$ with $x \le y$,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \frac{2\alpha}{1-\alpha}\|Tx - x\|^2 + \frac{2|\alpha|}{1-\alpha}\|Tx - x\|(\|x - y\| + \|Tx - Ty\|).$$

This completes the proof. □

Definition 2.3. A Banach space E is said to be:

(1) *strictly convex* if

$$\left\|\frac{x+y}{2}\right\| < 1$$

for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$;
(2) *uniformly convex* if, for all $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that

$$\frac{\|x+y\|}{2} < 1 - \delta$$

for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \ge \varepsilon$.

The following inequality was showed by Xu [18] in a uniformly convex Banach space E, which is known as *Xu's inequality*.

Lemma 2.4 (Xu [18, Theorem 2]). For any real numbers $q > 1$ and $r > 0$, a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $f: [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$ such that

$$\|tx + (1-t)y\|^q \le t\|x\|^q + (1-t)\|y\|^q - \omega(q,t)f(\|x-y\|) \tag{2.3}$$

for all $x, y \in B_r(0) = \{x \in E : \|x\| \le r\}$ and $t \in [0, 1]$, where $\omega(q,t) = t^q(1-t) + t(1-t)^q$. In particular, taking $q = 2$ and $t = \frac{1}{2}$,

$$\left\|\frac{x+y}{2}\right\|^2 \le \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{4}f(\|x-y\|). \tag{2.4}$$

The following conclusion is well known.

Lemma 2.5 (Takahashi [17, Theorem 1.3.11]). Let C be a nonempty closed convex subset of a reflexive Banach space E. Assume that g: $C \to R$ is a proper convex lower semi-continuous and coercive function (i.e., $\lim_{\|x\|\to\infty} f(x) = \infty$). Then the function g attains its minimum on C, that is, there exists $x \in C$ such that

$$g(x) = \inf_{y\in C} g(y).$$

3. Fixed points of monotone α-nonexpansive mappings

In this section, for a monotone α-nonexpansive mapping T, we consider the Mann iteration defined by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n \tag{3.1}$$

for each $n \ge 1$, where $\{\beta_n\}$ is a sequence in (0, 1). In the sequel, we denote

$$F_{\le}(T) = \{p \in F(T) : p \le x_1\} \quad \text{and} \quad F_{\ge}(T) = \{p \in F(T) : x_1 \le p\}.$$

Note that, since the partial order $\le$ is defined by the closed convex cone P, it is obvious that both $F_{\le}(T)$ and $F_{\ge}(T)$ are closed convex.

The following lemma is showed by Dehaish and Khamsi [2], where the conclusion (2) is obtained from the proof of [2, Lemma 3.1].

Lemma 3.1 (Dehaish and Khamsi [2, Lemma 3.1]). *Let K be a nonempty closed convex subset of an ordered Banach space* (E, $\le$) *and T: $K \to K$ be a monotone mapping. Assume that the sequence $\{x_n\}$ is defined by the Mann iteration (3.1) and $x_1 \le Tx_1$ (or $Tx_1 \le x_1$). Then we have*

(1) $x_n \le x_{n+1} \le Tx_n \le Tx_{n+1}$ *(or* $Tx_{n+1} \le Tx_n \le x_{n+1} \le x_n$*);*
(2) $x_n \le x$ *(or* $x \le x_n$*) for all* $n \ge 1$ *provided* $\{x_n\}$ *weakly converges to a point* $x \in K$.

In this section, we prove some existence theorems of fixed points of monotone α-nonexpansive mappings in a uniformly convex ordered Banach space (E, $\le$).

Theorem 3.2. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space* (E, $\le$) *and T: $K \to K$ be a monotone α-nonexpansive mapping. Assume that $x_1 \le Tx_1$ and the sequence $\{x_n\}$ defined by the Mann iteration (3.1) is bounded with $x_n \le y$ for some $y \in K$ and*

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Then $F_{\ge}(T) \ne \emptyset$.

Proof. It follows from $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$ that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

The Lemma 3.1 implies that $x_1 \le x_{n_k} \le x_{n_{k+1}} \le x_{n_{k+1}}$. Let $C_k = \{z \in K : x_{n_k} \le z\}$ for all $k \ge 1$. Clearly, for each $k \ge 1$, C_k is closed convex and $y \in C_k$ and so C_k is nonempty too. Let $C = \bigcap_{k=1}^{\infty} C_k$. Then C is a nonempty closed convex subset of K ($y \in C$). By the boundedness of $\{x_{n_k}\}$, a function $g: C \to [0, +\infty)$ may be defined by

$$g(z) = \limsup_{k\to\infty} \|x_{n_k} - z\|^2$$

for all $z \in C$. It follows from Lemma 2.5 that there exists $z^* \in C$ such that

$$g(z^*) = \inf_{z\in C} g(z) \tag{3.2}$$

and, by the definition of C,

$$x_1 \le x_{n_1} \le x_{n_2} \le \cdots \le x_{n_k} \le x_{n_{k+1}} \le \cdots \le z^*.$$

It follows from the monotonicity of T and Lemma 3.1 that

$$x_{n_k} \leq Tx_{n_k} \leq Tz^*$$

for each $k \geq 1$, which means that $Tz^* \in C$ and hence $\frac{z^*+Tz^*}{2} \in C$. So, by the definition (3.2) of z^*, we have

$$g(z^*) \leq g\left(\frac{z^*+Tz^*}{2}\right) \quad \text{and} \quad g(z^*) \leq g(Tz^*). \tag{3.3}$$

On the other hand, it follows from Lemma 2.2 that

$$\|Tx_{n_k} - Tz^*\|^2 \leq \|x_{n_k} - z^*\|^2 + \frac{2\alpha}{1-\alpha}\|Tx_{n_k} - x_{n_k}\|^2 + \frac{2|\alpha|}{1-\alpha}\|Tx_{n_k} - x_{n_k}\|(\|x_{n_k} - z^*\| + \|Tx_{n_k} - Tz^*\|).$$

By the boundedness of the sequence $\{x_n\}$ and $\lim_{k\to\infty}\|x_{n_k} - Tx_{n_k}\| = 0$, we have

$$\limsup_{k\to\infty}\|Tx_{n_k} - Tz^*\|^2 \leq \limsup_{k\to\infty}\|x_{n_k} - z^*\|^2 \tag{3.4}$$

and so

$$\begin{aligned} g(Tz^*) &= \limsup_{k\to\infty}\|x_{n_k} - Tz^*\|^2 \\ &= \limsup_{k\to\infty}\|Tx_{n_k} - Tz^*\|^2 \\ &\leq \limsup_{k\to\infty}\|x_{n_k} - z^*\|^2 \\ &= g(z^*). \end{aligned} \tag{3.5}$$

Now, we show $z^* = Tz^*$. In fact, it follows from Lemma 2.4 ($q = 2$ and $t = \frac{1}{2}$) and (3.5) that

$$\begin{aligned} g\left(\frac{z^*+Tz^*}{2}\right) &= \limsup_{k\to\infty}\left\|x_{n_k} - \frac{z^*+Tz^*}{2}\right\|^2 \\ &= \limsup_{k\to\infty}\left\|\frac{x_{n_k}-z^*}{2} + \frac{x_{n_k}-Tz^*}{2}\right\|^2 \\ &\leq \limsup_{k\to\infty}\left(\frac{1}{2}\|x_{n_k}-z^*\|^2 + \frac{1}{2}\|x_{n_k}-Tz^*\|^2 - \frac{1}{4}f(\|z^*-Tz^*\|)\right) \\ &\leq \frac{1}{2}g(z^*) + \frac{1}{2}g(Tz^*) - \frac{1}{4}f(\|z^*-Tz^*\|) \\ &\leq g(z^*) - \frac{1}{4}f(\|z^*-Tz^*\|). \end{aligned}$$

Noticing (3.3), we have

$$\frac{1}{4}f(\|z^*-Tz^*\|) \leq g(z^*) - g\left(\frac{z^*+Tz^*}{2}\right) \leq 0$$

and hence $f(\|z^*-Tz^*\|) = 0$. So $z^* = Tz^*$ by the property of f. This completes the proof. □

Using the same proof technique of Theorem 3.2 and putting $C_k = \{z \in K : z \leq x_{n_k}\}$ and $x_{n+1} \leq x_n$, we can prove the following theorem.

Theorem 3.3. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and T: $K \to K$ be a monotone α-nonexpansive mapping. Assume that $x_1 \geq Tx_1$ and the sequence $\{x_n\}$ defined by the Mann iteration (3.1) is bounded with $x_n \geq y$ for some $y \in K$ and*

$$\liminf_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Then $F_{\leq}(T) \neq \emptyset$.

Theorem 3.4. *Let $(E, \leq)$ be a uniformly convex ordered Banach space with the partial order $\leq$ with respect to closed convex cone P and T: $P \to P$ be a monotone α-nonexpansive mapping. Assume that $x_1 = 0$ and the sequence $\{x_n\}$ defined by the Mann iteration (3.1) is bounded with $x_n \leq y$ for some $y \in P$ and*

$$\liminf_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Then $F_{\geq}(T) \neq \emptyset$.

Proof. It follows from the definition of the partial order $\leq$ that $x_1 = 0 \leq T0 = Tx_1$. Then the conclusions directly follow from Theorem 3.2. □

4. The convergence of the Mann iteration

In this section, we consider the convergence of the Mann iteration for a monotone α-nonexpansive mapping T in an ordered Banach space $(E, \preceq)$.

Theorem 4.1. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \preceq)$ and $T: K \to K$ be a monotone α-nonexpansive mapping. Assume that the sequence $\{x_n\}$ defined by the Mann iteration (3.1) with $x_1 \preceq Tx_1$ (or $Tx_1 \preceq x_1$) and $F_{\succeq}(T) \neq \emptyset$ (or $F_{\preceq}(T) \neq \emptyset$). Then we have*

(1) *the sequence $\{x_n\}$ is bounded;*
(2) *$\|x_{n+1} - p\| \le \|x_n - p\|$ and the limit $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F_{\succeq}(T)$ (or $p \in F_{\preceq}(T)$);*
(3) *$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$ provided $\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0$;*
(4) *$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ provided $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.*

Proof. Without loss of generality, we assume that $x_1 \preceq p \in F_{\succeq}(T)$. We claim $x_n \preceq p$ for all $n \ge 1$. In fact, since T is monotone, we have

$$x_1 \preceq Tx_1 \preceq Tp = p$$

and so

$$x_1 \preceq x_2 \preceq Tx_1 \preceq p.$$

Suppose that $x_n \preceq p$. Then $Tx_n \preceq Tp = p$. It follows from Lemma 3.1 that

$$x_n \preceq x_{n+1} \preceq Tx_n \preceq p.$$

That is, $x_{n+1} \preceq p$, as claimed.

On the other hand, it follows from the definition of Lemma 2.2 that

$$\|Tx_n - p\| \le \|x_n - p\|$$

for all $n \ge 1$ and so

$$\begin{aligned}\|x_{n+1} - p\| &\le \beta_n \|x_n - p\| + (1-\beta_n)\|Tx_n - p\| \\ &\le \beta_n \|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &= \|x_n - p\| \\ &\quad \cdots \\ &\le \|x_1 - p\|.\end{aligned}$$

Then the sequence $\{\|x_n - p\|\}$ is nonincreasing and bounded and hence (1) and (2) hold.

Now, we show (3) and (4). In fact, it follows from Lemma 2.4 ($q = 2$ and $t = \beta_n$) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\|^2 \\ &\le \beta_n \|x_n - p\|^2 + (1-\beta_n)\|Tx_n - p\|^2 - \beta_n(1-\beta_n) f(\|x_n - Tx_n\|) \\ &\le \|x_n - p\|^2 - \beta_n(1-\beta_n) f(\|x_n - Tx_n\|),\end{aligned}$$

which implies that

$$\beta_n(1-\beta_n) f(\|x_n - Tx_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Then it follows from (2) that

$$\limsup_{n\to\infty} \beta_n(1-\beta_n) f(\|x_n - Tx_n\|) = 0.$$

(3) Since $\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0$ and

$$\left(\limsup_{n\to\infty} \beta_n(1-\beta_n)\right)\left(\liminf_{n\to\infty} f(\|x_n - Tx_n\|)\right) \le \limsup_{n\to\infty} \beta_n(1-\beta_n) f(\|x_n - Tx_n\|),$$

we have

$$\liminf_{n\to\infty} f(\|x_n - Tx_n\|) = 0,$$

which implies (3) by the properties of f.

(4) Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and

$$\left(\liminf_{n\to\infty} \beta_n(1-\beta_n)\right)\left(\limsup_{n\to\infty} f(\|x_n - Tx_n\|)\right) \le \limsup_{n\to\infty} \beta_n(1-\beta_n) f(\|x_n - Tx_n\|),$$

we have

$$\lim_{n\to\infty} f(\|x_n - Tx_n\|) = \limsup_{n\to\infty} f(\|x_n - Tx_n\|) = 0,$$

which implies (4) by the properties of f. This completes the proof. □

Recall that a Banach space E is said to satisfy *Opial's condition* ([13]) if a sequence $\{x_n\}$ with $\{x_n\}$ converges weakly to a point $x \in E$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Theorem 4.2. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and T: $K \to K$ be a monotone α-nonexpansive mapping. Assume that E satisfies Opial's condition and the sequence $\{x_n\}$ is defined by the Mann iteration (3.1) with $x_1 \le Tx_1$ (or $Tx_1 \le x_1$). If $F_{\ge}(T) \neq \emptyset$ (or $F_{\le}(T) \neq \emptyset$) and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then the sequence $\{x_n\}$ converges weakly to a fixed point z of T.*

Proof. It follows from Theorem 4.1 that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to a point $z \in K$. From Lemma 3.1, it follows that $x_1 \le x_{n_k} \le z$ (or $z \le x_{n_k} \le x_1$) for all $k \ge 1$.

On the other hand, Lemma 2.2 (2) means that

$$\|Tx_{n_k} - Tz\|^2 \le \|x_{n_k} - z\|^2 + \frac{2\alpha}{1-\alpha}\|Tx_{n_k} - x_{n_k}\|^2 + \frac{2|\alpha|}{1-\alpha}\|Tx_{n_k} - x_{n_k}\|(\|x_{n_k} - z\| + \|Tx_{n_k} - Tz\|).$$

By the boundedness of the sequence $\{x_n\}$ and $\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$, we have

$$\limsup_{k\to\infty} \|Tx_{n_k} - Tz\|^2 \le \limsup_{k\to\infty} \|x_{n_k} - z\|^2$$

and hence

$$\limsup_{k\to\infty} \|Tx_{n_k} - Tz\| \le \limsup_{k\to\infty} \|x_{n_k} - z\|. \tag{4.1}$$

Now, we prove $z = Tz$. In fact, suppose that $z \neq Tz$. Then, by (4.1) and Opial's condition, we have

$$\begin{aligned}\limsup_{k\to\infty} \|x_{n_k} - z\| &< \limsup_{k\to\infty} \|x_{n_k} - Tz\| \\ &\le \limsup_{k\to\infty}(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tz\|) \\ &\le \limsup_{k\to\infty} \|x_{n_k} - z\|,\end{aligned}$$

which is a contradiction. This implies that $z \in F_{\ge}(T)$ (or $z \in F_{\le}(T)$). Using Theorem 4.1 (2), the limit $\lim_{n\to\infty} \|x_n - z\|$ exists.

Now, we show that the sequence $\{x_n\}$ converges weakly to the point z. Suppose that this does not hold. Then there exists a subsequence $\{x_{n_j}\}$ to converge weakly to a point $x \in K$ and $z \neq x$. Similarly, we must have $x = Tx$ and $\lim_{n\to\infty} \|x_n - x\|$ exists. It follows from Opial's condition that

$$\lim_{n\to\infty} \|x_n - z\| < \lim_{n\to\infty} \|x_n - x\| = \limsup_{j\to\infty} \|x_{n_j} - x\| < \lim_{n\to\infty} \|x_n - z\|.$$

This is a contradiction and hence $x = z$. This completes the proof. □

Theorem 4.3. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and T: $K \to K$ be a monotone α-nonexpansive mapping. Assume that the sequence $\{x_n\}$ is defined by the Mann iteration (3.1) with $x_1 \le Tx_1$. If $\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $y \in F_{\ge}(T)$.*

Proof. Following the compactness of K, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to a point $y \in K$. From Lemma 3.1, it follows that $x_1 \le x_{n_k} \le y$ for all $k \ge 1$. By Theorem 3.2, we have $F_{\ge}(T) \neq \emptyset$. It follows from Theorem 4.1 that $\{x_n\}$ is bounded and

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Without loss of generality, we can assume that

$$\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

On the other hand, the Lemma 2.2 (2) guarantees that

$$\|Tx_{n_k} - Ty\|^2 \le \|x_{n_k} - y\|^2 + \frac{2\alpha}{1-\alpha}\|Tx_{n_k} - x_{n_k}\|^2 + \frac{2|\alpha|}{1-\alpha}\|Tx_{n_k} - x_{n_k}\|(\|x_{n_k} - y\| + \|Tx_{n_k} - Ty\|).$$

By the boundedness of the sequence $\{x_{n_k}\}$, $\lim_{k\to\infty}\|x_{n_k} - y\| = 0$ and $\lim_{k\to\infty}\|x_{n_k} - Tx_{n_k}\| = 0$, we have

$$\limsup_{k\to\infty} \|Tx_{n_k} - Ty\|^2 \le 0$$

and hence

$$\lim_{k\to\infty} \|Tx_{n_k} - Ty\| = 0. \tag{4.2}$$

Therefore, we have

$$\limsup_{k\to\infty} \|x_{n_k} - Ty\| \le \limsup_{k\to\infty}(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Ty\|) = 0$$

and so $\lim_{k\to\infty}\|x_{n_k} - Ty\| = 0$, which implies that $y \in F_{\ge}(T)$. Using Theorem 4.1 (2), the limit $\lim_{n\to\infty}\|x_n - y\|$ exists and so $\lim_{n\to\infty}\|x_n - y\| = 0$. This completes the proof. □

Similarly, the following theorems can be proved.

Theorem 4.4. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and T: $K \to K$ be a monotone α-nonexpansive mapping. Assume that the sequence $\{x_n\}$ is defined by the Mann iteration (3.1) with $x_1 \le Tx_1$. If $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $y \in F_{\ge}(T)$.*

Theorem 4.5. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and T: $K \to K$ be a monotone α-nonexpansive mapping. Assume that the sequence $\{x_n\}$ is defined by the Mann iteration (3.1) with $Tx_1 \le x_1$. If either $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$ or $\limsup_{n\to\infty}\beta_n(1-\beta_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $y \in F_{\le}(T)$.*

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