

# Solutions of the Fractional Logistics Equations via the Residual Power Series Method with Adomian Polynomials

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## Abstract

In this article, a combined form of the residual power series method with the Adomian polynomial is developed for analytic treatment of the fractional logistic equations. The Caputo operator is used to define the derivative of fractional order. The convergent analysis of solution is proposed. Illustrative examples are examined to support the proposed analysis. The fractional order solutions are compared to the integer order solutions.

## 1 Introduction

Fractional differential equations are generalizations of differential equations from integer order to non-integer order. The idea about the half order deriva-

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tive is originated Leibniz and L'Hôpital in 1665 [20]. The topic of fractional calculus has recently considered to be valuable tools in the modeling of many phenomena in real world. Nowadays, various definitions of fractional derivative have arisen. However, the definitions that frequently appeared in most literatures are of Riemann-Liouville type and Caputo type. Many researchers are in an attempt to develop a reliable method to solve fractional differential equations (FDEs), see for example [17, 18]. Nonetheless, all existing computational tools make the computation so cumbersome. Among those tools, residual power series method and Adomian decomposition method will be discussed here.

Residual power series method (RPSM) was proposed in [1]. The solution is written in the form of fractional power series. By means of truncating the series into first terms, the RPSM uses the  $n$ -th residual function in determining the  $n$ -th coefficients of the fractional power series for iteratively. Many consequence research papers indicate that the RPSM quite suits to nonlinear fractional differential equations with variable coefficients [3, 15, 21, 19, 5, 4, 16].

After introduced by George Adomian in 1980s, the Adomian decomposition method (ADM) becomes an effective tool which researchers commonly used to solve nonlinear differential equations. The key idea of the method is to express the solution in the form of infinite series. The nonlinear term is handled by an infinite series of special polynomials called Adomian polynomials [2]. The Adomian polynomials can be well combined with other series solution method such as differential transformation (DTM), homotopy analysis method (HAM) [7], and variational iteration method (VIM) [9]. The applications of the ADM to fractional differential equations can be found in [6] and the references there in.

Recently theory of fractional differential equations were tackled by many researchers. For the existence of solutions for FDEs, one can see [10, 11, 12, 13, 14] and the references therein. The existence of solutions are the most important qualitative properties of FDEs. The study of the existence of solutions ensures the essential conditions required for a solution of fractional differential equations.

The basic motivation of present study is the extension of RPSM equipped with the Adomian polynomials to tackle the fractional logistic equations. We first introduce some definitions and notations used throughout in this article in Section 2. In Section 3, we show the advantage of integrating the Adomian polynomials to the RPSM in reducing some computational work on algebraic manipulation. The convergence analysis of the proposed scheme

is discussed. In Section 4, the nonlinear fractional logistics equation are solved to illustrate the efficiency, applicability and simplicity of RPSM via the Adomian polynomial. The numerical errors versus the exact solutions are also reported for  $\alpha = 1$ .

## 2 Preliminaries and Notations

**Definition 2.1.** Let  $n$  be the smallest integer greater than or equal to  $\alpha$ . The Caputo fractional derivative of order  $\alpha > 0$  is defined as

$$D_t^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(\tau) d\tau, & n-1 < \alpha < n \\ u^{(n)}(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

where  $\Gamma(\cdot)$  denotes the well-known Gamma function.

**Theorem 2.2.** The Caputo fractional derivative of the power function is given by

$$D_t^\alpha t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, & n-1 < \alpha < n, p > n-1, p \in \mathbb{R} \\ 0, & n-1 < \alpha < n, p \leq n-1, p \in \mathbb{N}. \end{cases}$$

**Definition 2.3.** A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha} = c_0 + c_1 (t-t_0)^\alpha + c_2 (t-t_0)^{2\alpha} + \dots$$

where  $0 \leq n-1 < \alpha$ ,  $t \geq t_0$ , is called fractional power series (FPS) about  $t = t_0$ .

**Theorem 2.4.** Suppose that  $f$  has a fractional power series represent at  $t = t_0$  of the form

$$f(t) = \sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha}, \quad 0 \leq n-1 < \alpha, \quad t_0 \leq t < t_0 + R,$$

where  $R$  is the radius of convergence. If  $D^{m\alpha} f(t)$ ,  $m = 0, 1, 2, \dots$  are continuous on  $(t_0, t_0 + R)$ , then  $c_m = \frac{D^{m\alpha} f(t_0)}{\Gamma(1+m\alpha)}$ .

### 3 Algorithm of RPSM with Adomian Polynomials

Consider the fractional differential equation

$$D_t^\alpha u(t) = L(u(t)) + F(u(t)), \quad 0 < t < T, \quad (3.1)$$

subject to the initial condition

$$u(0) = u_0, \quad (3.2)$$

where  $L(u(t))$  and  $F(u(t))$  are linear term and nonlinear term respectively.  $D_t^\alpha$  means the Caputo fractional derivative with respect to  $t$  of order  $0 < \alpha < 1$ .

The RPSM assumes the solution  $u(t)$ , of Eq. (3.1), in a form of fractional power series centered at  $t = 0$  as

$$u(t) = \sum_{n=0}^{\infty} \frac{u_n t^{n\alpha}}{\Gamma(1 + n\alpha)}. \quad (3.3)$$

Using the initial condition in Eq. (3.2), we approximate  $u(t)$  in Eq. (3.3) by

$$u_k(t) = u_0 + \sum_{n=1}^k \frac{u_n t^{n\alpha}}{\Gamma(1 + n\alpha)}, \quad k = 1, 2, 3, \dots \quad (3.4)$$

From Eq. (3.4), the  $k^{\text{th}}$  residual power series approximation  $u_k(t)$  will be obtained by computing the component  $u_1, u_2, \dots, u_k$ . Before computing these components, we define the  $k$ th residual function

$$Res_k(t) = D_t^\alpha u_k(t) - L(u_k(t)) - F(u_k(t)). \quad (3.5)$$

To find the coefficients  $u_1, u_2, \dots, u_k$  of the RPS solution (3.4), we solve equation

$$D_t^{(k-1)\alpha} Res_k(0) = 0, \quad k = 1, 2, 3, \dots \quad (3.6)$$

As mentioned above, the RPSM has been used for nonlinear differential equation including the nonlinear fractional differential equation. In this work, the Adomian polynomials are used for the nonlinear term in an easy way. Adomian polynomials decompose a function  $u_k(t)$  into a sum of components

$$u_k(t) = \sum_{n=0}^{\infty} v_n(t).$$

A nonlinear operator  $F$  of  $u(t)$  can be written in the form of

$$F(u(t)) = \sum_{n=0}^{\infty} A_n,$$

where  $A_n$  are known as the Adomian polynomials determined formally from the relation

$$A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left[ F \left( \sum_{i=0}^{\infty} \lambda^i v_i \right) \right] \right]_{\lambda=0}.$$

Then, the first few polynomials are given by

$$\begin{aligned} A_0 &= F(v_0), \\ A_1 &= v_1 F'(v_0), \\ A_2 &= v_2 F'(v_0) + \frac{1}{2!} v_1^2 F''(v_0), \\ A_3 &= v_3 F'(v_0) + v_1 v_2 F''(v_0) + \frac{1}{3!} v_1^3 F'''(v_0), \\ A_4 &= v_4 F'(v_0) + \left( \frac{1}{2!} v_2^2 + v_1 v_3 \right) F''(v_0) + \frac{1}{2!} v_1^2 v_2 F'''(v_0) + \frac{1}{4!} v_1^4 F''''(v_0). \end{aligned}$$

Other polynomials can be calculated in similar manner.

## 4 Applications of RPSM to Fractional Logistics Equation

Consider the fractional logistic equation

$$D_t^\alpha u(t) = \rho^\alpha u(1-u), \quad \alpha \in (0, 1], \quad (4.7)$$

subject to the initial condition

$$u(0) = u_0, \quad u_0 > 0, \quad (4.8)$$

and  $\rho > 0$ . The derivative in fractional logistic Eq. (4.7) is in the Caputo sense. According to the RPSM, let  $u(t)$  be the solution of fractional logistic equation as a fractional power series about  $t = 0$  of the form

$$u(t) = \sum_{n=0}^{\infty} \frac{u_n t^{n\alpha}}{\Gamma(1+n\alpha)}. \quad (4.9)$$

Using the initial condition (4.8), we approximate  $u(t)$  in Eq. (4.9) by

$$u_k(t) = u_0 + \sum_{n=1}^k \frac{u_n t^{n\alpha}}{\Gamma(1 + n\alpha)}, \quad k = 1, 2, 3, \dots \quad (4.10)$$

To find the values of the RPS coefficient  $u_n$ ,  $n = 1, 2, 3, \dots$ , we solve the equation

$$D_t^{(n-1)\alpha} Res_n(0) = 0, \quad n = 1, 2, 3, \dots, \quad (4.11)$$

where  $Res_k(t)$  is the  $k$ th residual function and it defined by

$$Res_k(t) = D_t^\alpha u_k(t) - \rho^\alpha (u_k(t) - u_k^2(t)). \quad (4.12)$$

Since the fractional logistic equation (4.7) is a nonlinear fractional differential equation in term of  $u^2(t)$ , Adomian polynomials are implemented to calculate nonlinear term of  $u^2(t)$ . So, the Adomian polynomials and the residual power series method are combined to solve the fractional logistic equation.

We first set

$$u_k(t) = \sum_{i=0}^k v_i, \quad (4.13)$$

where  $v_0 = u_0$  and

$$v_i = \frac{u_i t^{i\alpha}}{\Gamma(1 + i\alpha)}, \quad i = 1, 2, 3, \dots, k. \quad (4.14)$$

Since  $F(u_k(t))$  is the nonlinear operator, let

$$F(u_k(t)) = \sum_{n=0}^{\infty} A_n, \quad (4.15)$$

where  $A_n$  are called Adomian polynomials determined formally from the relation

$$A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left[ F \left( \sum_{i=0}^k \lambda^i v_i \right) \right] \right]_{\lambda=0}. \quad (4.16)$$

From Eq. (4.13), we can rewritten the nonlinear polynomials  $u_k^2(t)$  as

$$F(u_k(t)) = (v_0 + v_1 + v_2 + v_3 + \dots + v_k)^2 = \sum_{n=0}^{\infty} A_n.$$

Consequently, the Adomian polynomials for  $F(u_k(t)) = u_k^2(t)$  are given by

$$\begin{aligned}
 A_0 &= v_0^2 \\
 A_1 &= 2v_0v_1 \\
 A_2 &= 2v_0v_2 + v_1^2 \\
 A_3 &= 2v_0v_3 + 2v_1v_2 \\
 A_4 &= v_2^2 + 2v_1v_3 + 2v_0v_4 \\
 A_5 &= 2v_2v_3 + 2v_0v_5 + 2v_1v_4 \\
 A_6 &= 2v_0v_6 + 2v_1v_5 + 2v_2v_4 + v_3^2 \\
 A_7 &= 2v_0v_7 + 2v_2v_5 + 2v_3v_4 + 2v_1v_6 \\
 A_8 &= 2v_2v_6 + 2v_3v_5 + v_4^2 + 2v_0v_8 + 2v_1v_7.
 \end{aligned}$$

Other polynomials can be calculated by Eq. (4.16).

To find  $u_1$ , we substitute the first RPS approximate solution

$$u_1(t) = u_0 + u_1 \frac{t^\alpha}{\Gamma(1 + \alpha)}$$

into Eq. (4.12) as follows

$$\begin{aligned}
 Res_1(t) &= D_t^\alpha u_1(t) - \rho^\alpha (u_1(t) - u_1^2(t)) \\
 &= D_t^\alpha \left( u_0 + u_1 \frac{t^\alpha}{\Gamma(1 + \alpha)} \right) - \rho^\alpha \left( u_0 + u_1 \frac{t^\alpha}{\Gamma(1 + \alpha)} \right) \\
 &\quad + \rho^\alpha \left( u_0 + u_1 \frac{t^\alpha}{\Gamma(1 + \alpha)} \right)^2 \\
 &= u_1 - \rho^\alpha \left( u_0 + u_1 \frac{t^\alpha}{\Gamma(1 + \alpha)} \right) + \rho^\alpha \left( u_0 + u_1 \frac{t^\alpha}{\Gamma(1 + \alpha)} \right)^2.
 \end{aligned}$$

Then, we solve  $Res_1(0) = 0$  to get

$$u_1 = \rho^\alpha (u_0 - u_0^2). \quad (4.17)$$

To find  $u_2$  in Eq. (4.10), the second RPS approximate solution is in form

$$u_2(t) = u_0 + u_1 \frac{t^\alpha}{\Gamma(1 + \alpha)} + u_2 \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)}. \quad (4.18)$$

Using Adomian polynomials and  $F(u_2(t)) = u_2^2(t)$ , we have

$$\begin{aligned}
 F(u_2(t)) &= (v_0 + v_1 + v_2)^2 \\
 &= \sum_{n=0}^{\infty} A_n \\
 &= v_0^2 + 2v_0v_1 + 2v_0v_2 + v_1^2 + 2v_1v_2 + v_2^2.
 \end{aligned}$$

From  $v_0 = u_0$  and Eq. (4.14), we have

$$u_2^2(t) = F(u_2(t)) = u_0^2 + 2u_0u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + 2u_0u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\frac{t^\alpha}{\Gamma(1+\alpha)}\right)^2 + 2u_1u_2\frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2. \quad (4.19)$$

Substituting Eq.4.18) and Eq. (4.19) into Eq. (4.12) yields

$$\begin{aligned} Res_2(t) &= D_t^\alpha u_2(t) - \rho^\alpha (u_2(t) - u_2^2(t)) \\ &= \left(u_1 + u_2\frac{t^\alpha}{\Gamma(1+\alpha)}\right) - \rho^\alpha \left(u_0 + u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) \\ &\quad + \rho^\alpha \left[ u_0^2 + 2u_0u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + 2u_0u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\frac{t^\alpha}{\Gamma(1+\alpha)}\right)^2 \right. \\ &\quad \left. + 2u_1u_2\frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2 \right]. \end{aligned}$$

Applying  $D_t^\alpha$  on both sides of Eq. (4.20), we obtain

$$\begin{aligned} D_t^\alpha Res_2(t) &= u_2 - \rho^\alpha \left(u_1 + u_2\frac{t^\alpha}{\Gamma(1+\alpha)}\right) \\ &\quad + \rho^\alpha \left[ 2u_0u_1 + 2u_0u_2\frac{t^\alpha}{\Gamma(1+\alpha)} + u_1^2\frac{\Gamma(1+2\alpha)t^\alpha}{\Gamma^3(1+\alpha)} \right. \\ &\quad \left. + 2u_1u_2\frac{\Gamma(1+3\alpha)t^{2\alpha}}{\Gamma(1+\alpha)\Gamma^2(1+2\alpha)} + u_2^2\frac{\Gamma(1+4\alpha)t^{3\alpha}}{\Gamma^2(1+2\alpha)\Gamma(1+3\alpha)} \right]. \end{aligned}$$

Then, we solve  $D_t^\alpha Res_2(0) = 0$  to get

$$D_t^\alpha Res_2(0) = u_2 - \rho^\alpha (u_1 - 2u_0u_1) = 0.$$

We have the coefficient  $u_2$  as

$$u_2 = \rho^\alpha (u_1 - 2u_0u_1). \quad (4.20)$$

To find  $u_3$ , the third RPS approximate solution is in form

$$u_3(t) = u_0 + u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}. \quad (4.21)$$

By using Adomian polynomials and  $u_3^2(t) = F(u_3(t))$ , we have

$$\begin{aligned} F(u_3(t)) &= (v_0 + v_1 + v_2 + v_3)^2 \\ &= \sum_{n=0}^{\infty} A_n \\ &= v_0^2 + 2v_0v_1 + 2v_0v_2 + v_1^2 + 2v_0v_3 + 2v_1v_2 + v_2^2 + 2v_1v_3 \\ &\quad + 2v_2v_3 + v_3^2. \end{aligned}$$



From  $v_0 = u_0$  and Eq. (4.14), we have

$$\begin{aligned}
 u_3^2(t) &= F(u_3(t)) \\
 &= u_0^2 + 2u_0u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + 2u_0u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\frac{t^\alpha}{\Gamma(1+\alpha)}\right)^2 \\
 &\quad + 2u_0u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 2u_1u_2\frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2 \\
 &\quad + 2u_1u_3\frac{t^{4\alpha}}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} + 2u_2u_3\frac{t^{5\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)} + \left(u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)^2.
 \end{aligned} \tag{4.22}$$

Substituting Eq. (4.21) and Eq. (4.22) into Eq. (4.12) gives

$$\begin{aligned}
 Res_3(t) &= D_t^\alpha u_3(t) - \rho^\alpha (u_3(t) - u_3^2(t)) \\
 &= \left(u_1 + u_2\frac{t^\alpha}{\Gamma(1+\alpha)} + u_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) \\
 &\quad - \rho^\alpha \left(u_0 + u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right) \\
 &\quad + \rho^\alpha \left[ u_0^2 + 2u_0u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + 2u_0u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\frac{t^\alpha}{\Gamma(1+\alpha)}\right)^2 \right. \\
 &\quad + 2u_0u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 2u_1u_2\frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2 \\
 &\quad \left. + 2u_1u_3\frac{t^{4\alpha}}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} + 2u_2u_3\frac{t^{5\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)} + \left(u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)^2 \right].
 \end{aligned} \tag{4.23}$$

Then, we solve  $D_t^{2\alpha} Res_3(0) = 0$  to get

$$u_3 = \rho^\alpha \left( u_2 - 2u_0u_2 - u_1^2 \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)} \right). \tag{4.24}$$

To find  $u_4$ , the fourth RPS approximate solution is in form

$$u_4(t) = u_0 + u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + u_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}. \tag{4.25}$$

Using the Adomian polynomials and  $F(u_4(t)) = u_4^2(t)$ , we have

$$\begin{aligned}
 F(u_4(t)) &= (v_0 + v_1 + v_2 + v_3 + v_4)^2 \\
 &= \sum_{n=0}^{\infty} A_n \\
 &= v_0^2 + 2v_0v_1 + 2v_0v_2 + v_1^2 + 2v_0v_3 + 2v_1v_2 + v_2^2 + 2v_1v_3 + 2v_0v_4 \\
 &\quad + 2v_2v_3 + 2v_1v_4 + 2v_2v_4 + v_3^2 + 2v_3v_4 + v_4^2.
 \end{aligned}$$

From  $v_0 = u_0$  and Eq. (4.14), we have

$$\begin{aligned}
 u_4^2(t) &= u_0^2 + 2u_0u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + 2u_0u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\frac{t^\alpha}{\Gamma(1+\alpha)}\right)^2 \\
 &\quad + 2u_0u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 2u_1u_2\frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2 \\
 &\quad + 2u_1u_3\frac{t^{4\alpha}}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} + 2u_0u_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + 2u_2u_3\frac{t^{5\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)} \\
 &\quad + 2u_1u_4\frac{t^{5\alpha}}{\Gamma(1+\alpha)\Gamma(1+4\alpha)} + 2u_2u_4\frac{t^{6\alpha}}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)} + \left(u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)^2 \\
 &\quad + 2u_3u_4\frac{t^{7\alpha}}{\Gamma(1+3\alpha)\Gamma(1+4\alpha)} + \left(u_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right)^2.
 \end{aligned} \tag{4.26}$$

Proceeding as before, substituting Eq. (4.25) and Eq. (4.26) into Eq. (4.12) gives

$$\begin{aligned}
 Res_4(t) &= u_1 + u_2\frac{t^\alpha}{\Gamma(1+\alpha)} + u_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + u_4\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\
 &\quad - \rho^\alpha \left( u_0 + u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + u_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \right) \\
 &\quad + \rho^\alpha \left[ u_0^2 + 2u_0u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + 2u_0u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\frac{t^\alpha}{\Gamma(1+\alpha)}\right)^2 \right. \\
 &\quad + 2u_0u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 2u_1u_2\frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2 \\
 &\quad + 2u_1u_3\frac{t^{4\alpha}}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} + 2u_0u_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + 2u_2u_3\frac{t^{5\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)} \\
 &\quad + 2u_1u_4\frac{t^{5\alpha}}{\Gamma(1+\alpha)\Gamma(1+4\alpha)} + \left(u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)^2 + 2u_3u_4\frac{t^{7\alpha}}{\Gamma(1+3\alpha)\Gamma(1+4\alpha)} \\
 &\quad \left. + \left(u_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right)^2 \right].
 \end{aligned} \tag{4.27}$$

Thus, we solve  $D_t^{3\alpha}Res_4(0) = 0$  to get

$$u_4 = \rho^\alpha \left( u_3 - 2u_0u_3 - u_1u_2\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \right). \tag{4.28}$$

In general to find  $u_k$  in Eq. (4.10), we substitute  $k^{th}$  RPS approximate solution  $u_k$  into

$$Res_k(t) = D_t^\alpha u_k(t) - \rho^\alpha (u_k(t) - u_k^2(t)).$$

Then, we solve  $D^{(k-1)\alpha}Res_k(0) = 0$  to obtain

$$u_k = \rho^\alpha \left( u_{k-1} - \sum_{i=0}^{k-1} \frac{u_i u_{k-1-i} \Gamma(1+(k-1)\alpha)}{\Gamma(1+i\alpha)\Gamma(1+(k-1-i)\alpha)} \right). \tag{4.29}$$

### Convergence Analysis

Now, we prove the convergence of the RPSM. We start by the Lemma 4.1.

**Lemma 4.1.** *The classical power series (CPS)  $\sum_{n=0}^{\infty} u_n t^n$ ,  $-\infty < t < \infty$ , has a radius of convergence  $R$  if and only if the fractional power series (FPS)  $\sum_{n=0}^{\infty} u_n t^{n\alpha}$ ,  $t \geq 0$ , has a radius of convergence  $R^{\frac{1}{\alpha}}$ .*

*Proof.* See [8]. □

**Theorem 4.2.** *The fractional power series solution of fractional logistic equation (4.7) is expressed as*

$$u(t) = \sum_{n=0}^{\infty} \frac{u_n t^{n\alpha}}{\Gamma(1 + n\alpha)}$$

where the coefficients are defined in equation (4.29) has a positive radius of convergence.

*Proof.* From Eq. (4.29), we can see that

$$\begin{aligned} & \frac{|u_k|}{\Gamma(1 + k\alpha)} \\ & \leq |\rho^\alpha| \left( \frac{|u_{k-1}| + \left| \sum_{i=0}^{k-1} \frac{u_i u_{k-1-i} \Gamma(1 + (k-1)\alpha)}{\Gamma(1 + i\alpha) \Gamma(1 + (k-1-i)\alpha)} \right|}{\Gamma(1 + k\alpha)} \right) \\ & \leq |\rho^\alpha| \left( \frac{|u_{k-1}|}{\Gamma(1 + k\alpha)} + \sum_{i=0}^{k-1} \frac{|u_i| |u_{k-1-i}| \Gamma(1 + (k-1)\alpha)}{\Gamma(1 + i\alpha) \Gamma(1 + (k-1-i)\alpha) \Gamma(1 + k\alpha)} \right) \\ & \leq |\rho^\alpha| \left( \frac{|u_{k-1}|}{\Gamma(1 + k\alpha)} + \max_k \left\{ \frac{\Gamma(1 + (k-1)\alpha)}{\Gamma(1 + i\alpha) \Gamma(1 + (k-1-i)\alpha) \Gamma(1 + k\alpha)} \right\} \sum_{i=0}^{k-1} |u_i| |u_{k-1-i}| \right) \\ & = A |u_{k-1}| + B \sum_{i=0}^{k-1} |u_i| |u_{k-1-i}| \end{aligned}$$

where

$$A = \frac{|\rho^\alpha|}{\Gamma(1 + k\alpha)}, \quad B = \max_k \left\{ \frac{\Gamma(1 + (k-1)\alpha)}{\Gamma(1 + i\alpha) \Gamma(1 + (k-1-i)\alpha) \Gamma(1 + k\alpha)} \right\} |\rho^\alpha|.$$

Let

$$f(t) = \sum_{k=0}^{\infty} a_k t^k \quad (4.30)$$

where  $a_0 = |u_0|$  and

$$a_k = A a_{k-1} + B \sum_{i=0}^{k-1} a_i a_{k-1-i}, \quad k = 1, 2, \dots \quad (4.31)$$

be the classical power series.

Thus,

$$\begin{aligned} \omega = f(t) &= a_0 + t \sum_{k=0}^{\infty} a_{k+1} t^k \\ &= a_0 + t \left( A \sum_{k=0}^{\infty} a_k t^k + B \sum_{k=0}^{\infty} \left( \sum_{i=0}^k a_i a_{k-i} \right) t^k \right). \end{aligned}$$

Let

$$F(t, \omega) = \omega - a_0 - t(A\omega + B\omega^2). \quad (4.32)$$

Then

$$F_{\omega}(t, \omega) = 1 - t(A + 2B\omega).$$

Regarding at point  $(0, a_0)$ , the function  $F(t, \omega)$  is 0 and the partial derivative of the function  $F(t, \omega)$  with respect to  $\omega$  is 1. We can see that  $F(t, \omega)$  is an analytic function, so  $F(t, \omega)$  has continuous derivatives. By implicit function theorem, there is a neighborhood of  $(0, a_0)$  so that whenever  $t$  is sufficiently close to 0 there is a unique  $\omega$  so that  $F(t, \omega) = 0$ . Then,  $f(t)$  is an analytic function in the neighborhood of the point  $(0, a_0)$  of the  $(t, \omega)$ -plane with a positive radius of convergence. From Lemma 4.1, the series in Eq. (4.9) converges.  $\square$

## 5 Numerical Example

In this section, numerical applications of the fractional logistic differential equations are presented. We present three examples to show the efficiency of the RPSM. Comparison with the exact solution of Eq. (4.7) when  $\alpha = 1$  is reported in Table 1-3. Let

$$\text{error}(t) = |u_{\text{exact}}(t) - u_k(t)|, \quad t \geq 0, \quad (5.33)$$

where  $u_{exact}(t)$  is the exact closed form solution of Eq. (4.7) subject to initial condition (4.8), when  $\alpha = 1$ , given by

$$u(t) = \frac{u_0}{u_0 + (1 - u_0)e^{-\rho t}}, \quad t \geq 0, \quad (5.34)$$

where  $u_0$  is the initial state at the time  $t = 0$ . In Examples 5.1 and 5.2, we use  $k = 3$ . Then

$$u_3(t) = u_0 + u_1 \frac{t^\alpha}{\Gamma(1 + \alpha)} + u_2 \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} + u_3 \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)}, \quad (5.35)$$

where

$$\begin{aligned} u_1 &= \rho^\alpha (u_0 - u_0^2) \\ u_2 &= \rho^\alpha (u_1 - 2u_0u_1) \\ u_3 &= \rho^\alpha \left( u_2 - 2u_0u_2 - u_1^2 \frac{\Gamma(1 + 2\alpha)}{\Gamma^2(1 + \alpha)} \right). \end{aligned}$$

**Example 5.1.** Consider the following fractional logistic equation

$$D_t^\alpha u(t) = \frac{1}{2^\alpha} u(1 - u), \quad t > 0, \quad 0 < \alpha \leq 1,$$

subject to the initial condition

$$u(0) = 0.85.$$

In Table 1, the numerical error of the 3<sup>rd</sup> RPS is presented for  $\alpha = 1$ . The behaviors of approximate solutions are plotted for different values of  $\alpha$ , where  $\alpha = \{1.0, 0.75, 0.5\}$ , in Figure 1.

Table 1: Error when  $\alpha = 1$ .

$t$	exact solution	$u_3(t)$	error( $t$ )
0.00	0.8500000000000000	0.8500000000000000	0
0.02	0.851270542513367	0.851270542493750	$1.96169 \times 10^{-11}$
0.04	0.852532190262384	0.852532189950000	$3.12384 \times 10^{-10}$
0.06	0.853784973905210	0.853784972331250	$1.57396 \times 10^{-9}$
0.08	0.855028924550894	0.855028919600000	$4.95089 \times 10^{-9}$
0.10	0.856264073748411	0.856264061718750	$1.20297 \times 10^{-8}$

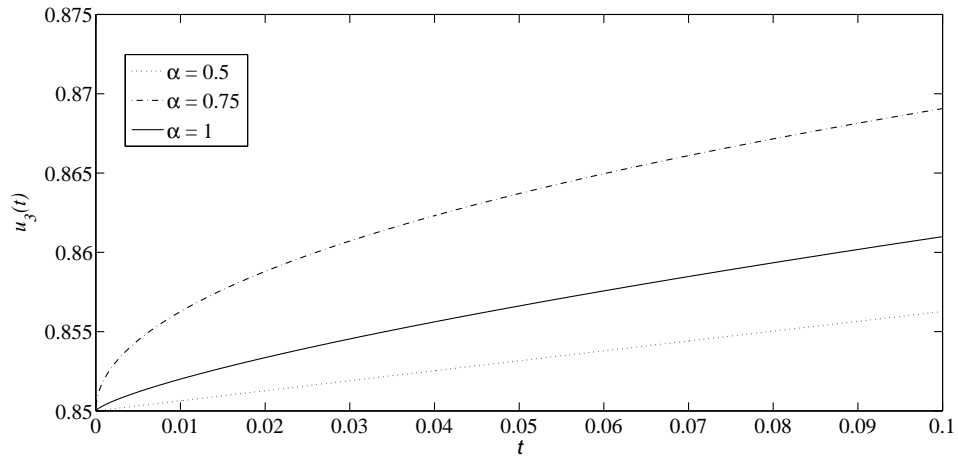


Figure 1: The approximate solution plots of Example 5.1 for different values of  $\alpha$ .

**Example 5.2.** Consider the following fractional logistic equation

$$D_t^\alpha u(t) = \frac{1}{4^\alpha} u(1 - u), \quad t > 0, \quad 0 < \alpha \leq 1.$$

subject to

$$u(0) = \frac{1}{3}.$$

The numerical error of the 3<sup>rd</sup> RPS, when  $\alpha = 1$ , is reported in Table 2. In Figure 2, the behavior of the 3<sup>rd</sup> RPS approximation is presented with different values of  $\alpha$ , where  $\alpha = \{1.0, 0.75, 0.5\}$ .

Table 2: Error when  $\alpha = 1$ .

$t$	exact solution	$u_3(t)$	error( $t$ )
0.00	0.3333333333333333	0.3333333333333333	0
0.02	0.334445368823947	0.334445368827160	$3.21300 \times 10^{-12}$
0.04	0.335559246862188	0.335559246913580	$5.13920 \times 10^{-11}$
0.06	0.336674958073288	0.3366749583333333	$2.60045 \times 10^{-10}$
0.08	0.337792493005687	0.337792493827160	$8.21437 \times 10^{-10}$
0.10	0.338911842131240	0.338911844135803	$2.00456 \times 10^{-9}$

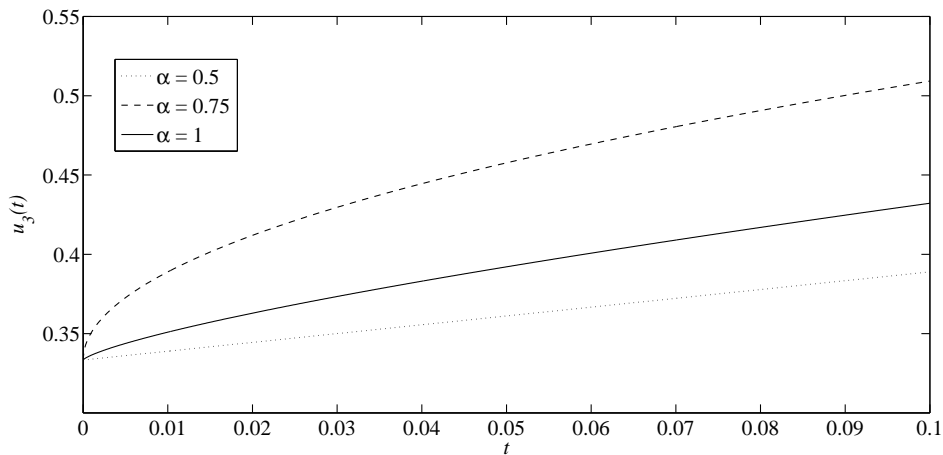


Figure 2: The approximate solution plots of Example 5.2 for different values of  $\alpha$ .

In Example 5.3, we use  $k = 4$ . Then, the approximate solution of fractional logistic equation is given by

$$u_4(t) = u_0 + u_1 \frac{t^\alpha}{\Gamma(1 + \alpha)} + u_2 \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} + u_3 \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)} + u_4 \frac{t^{4\alpha}}{\Gamma(1 + 4\alpha)}, \quad (5.36)$$

where

$$\begin{aligned} u_1 &= \rho^\alpha (u_0 - u_0^2) \\ u_2 &= \rho^\alpha (u_1 - 2u_0u_1) \\ u_3 &= \rho^\alpha \left( u_2 - 2u_0u_2 - u_1^2 \frac{\Gamma(1 + 2\alpha)}{\Gamma^2(1 + \alpha)} \right) \\ u_4 &= \rho^\alpha \left( u_3 - 2u_0u_3 - u_1u_2 \frac{\Gamma(1 + 3\alpha)}{\Gamma(1 + \alpha)\Gamma(1 + 2\alpha)} \right) \end{aligned}$$

Let

$$\text{error}(t) = |u_{\text{exact}}(t) - u_4(t)|, \quad t \geq 0. \quad (5.37)$$

**Example 5.3.** Consider the following fractional logistic equation

$$D_t^\alpha u(t) = \frac{1}{3^\alpha} u(1 - u), \quad t > 0, \quad 0 < \alpha \leq 1.$$

with the initial condition

$$u(0) = \frac{3}{4}.$$

Table 3 shows the numerical error of the 4<sup>th</sup> RPS when  $\alpha = 1$ . Figure 3 shows the representation of the 4<sup>th</sup> approximate solution with different values of  $\alpha$  such that  $\alpha = \{1.0, 0.75, 0.5\}$ .

Table 3: Error when  $\alpha = 1$ .

$t$	exact solution	$u_4(t)$	error( $t$ )
0.00	0.7500000000000000	0.7500000000000000	0
0.02	0.751247915518896	0.751247915514564	$4.33198 \times 10^{-12}$
0.04	0.752491657561459	0.752491657492284	$6.91750 \times 10^{-11}$
0.06	0.753731219529199	0.753731219179688	$3.49511 \times 10^{-10}$
0.08	0.754966595053057	0.754966593950617	$1.10244 \times 10^{-9}$
0.10	0.756197777992358	0.756197775306231	$2.68613 \times 10^{-9}$

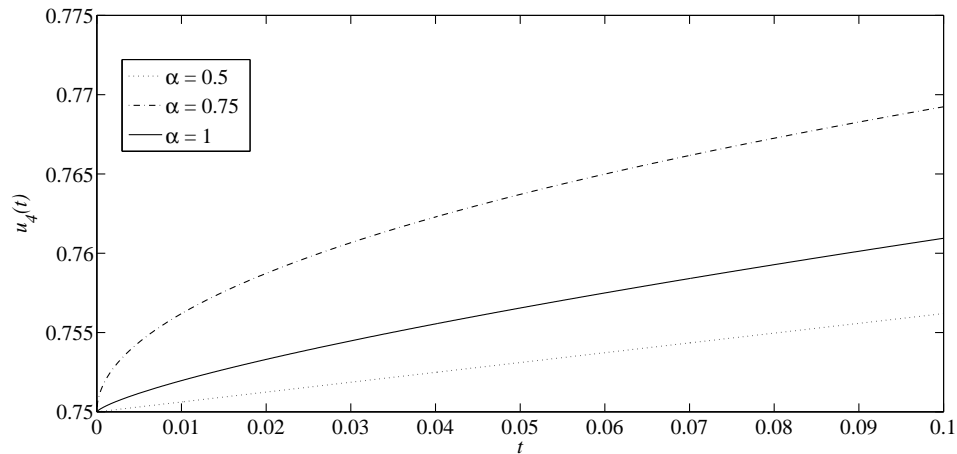


Figure 3: The approximate solution plots of Example 5.3 for different values of  $\alpha$ .

## 6 Conclusions

We have presented a combined form of the residual power series method with the Adomian polynomial for obtaining solutions of fractional logistic equations. We have applied the method to different examples. Comparison with the exact solution when  $\alpha = 1$  is reported. We obtained accurate results using a few terms. Therefore, we can see that the RPSM with Adomian



polynomial is very powerful for finding the approximate as well as analytical solution of many nonlinear models in science and engineering.

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