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A Non-Uniform Concentration Inequality for a Random Permutation Sum

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The purpose of this article is to give a non-uniform concentration inequality of a random permutation sum, $W_n = \sum_{i=1}^{n} Y(i, \pi(i))$, where $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ is a uniformly distributed random permutation of $1, 2, \ldots, n$ and $Y(i, j), i, j = 1, 2, \ldots, n$ are random variables such that $Y(i, j)'s$ and π all are stochastically independent. To do this, we assume the finiteness of third moment.

Keywords Concentration inequality; Non uniform bound; Random permutation sum.

Mathematics Subject Classification 60F05; 60G05.

1. Introduction and Main Result

Let $Y(i, j), i, j = 1, 2, \ldots, n$, be independent random variables with finite third moment and π be a random permutation on $\{1, 2, \ldots, n\}$ such that $Y(i, j)'$s and π all are stochastically independent. In this article, we concern with $W_n = \sum_{i=1}^{n} Y(i, \pi(i))$, which is called a random permutation sum. Several articles have mentioned the behavior of the limit of W_n. Von Bahr (1976) and Ho and Chen (1978) showed that, under some appropriate conditions, the distribution of W_n converges to the standard normal distribution. Bolthausen (1984) gave a Berry-Esseen-type bound by using an inductive approach of Stein's method. Neammanee and Suntornchost (2005) applied a concentration inequality approach to obtain the uniform rate of convergence under the finiteness of third moments. In that work, they used a uniform concentration inequality of W_n as an important tool. In this article, we apply the idea from Laipaporn and Neammanee (2006) and Laipaporn and Sungkamongkol (2009) to give a non-uniform concentration inequality of W_n.

Let X be a random variable. The function $Q_X : [0, \infty) \to \mathbb{R}$ defined by

$$Q_X(\lambda) = \sup_x P(x \le X \le x + \lambda)$$

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is called a **uniform (Lévy) concentration function** of X and the function $Q_X : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ defined by

$$Q_X(x; \lambda) = P(x \leq X \leq x + \lambda)$$

is called a **non-uniform (Lévy) concentration function** of X.

Throughout this article, we assume that $VarW_n = 1$,

$$\begin{aligned} &E|Y(i, j)|^3 \leq \infty, \quad 1 \leq i, \ \ j \leq n, \\ &\sum_{i_j=1}^{n} EY(i_1, i_2) = 0 \quad \text{for } j = 1, 2, \end{aligned} \tag{1.1}$$

and let

$$\delta_3 = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} E|Y(i, j)|^3.$$

Neammanee and Suntornchost (2005) gave a uniform concentration inequality as follows.

Theorem 1.1. *If $\delta_3 < \frac{1}{350}$ and $n \geq 32$, then*

$$P(a \leq W_n \leq a + \lambda) \leq 2\lambda + 78\delta_3$$

for any real numbers a, λ with $\lambda \geq 0$.

This article gives a non-uniform concentration inequality of W_n.

Theorem 1.2. *For $n \geq 32$, then*

$$\begin{aligned} P(z \leq W_n \leq z + \lambda) \leq{} & \frac{(189.3936 + 79.6348n\delta_3^2)\lambda}{(1+z)^3} + \frac{(400.7874 + 159.2696n\delta_3^2)\delta_3}{(1+z)^2} \\ & + \frac{48.29}{\sqrt{n}(1+z)^2} + \frac{C\delta_3^{\frac{3}{4}}(1 + n\delta_3^2)^{\frac{1}{4}}}{n^{\frac{5}{12}}} + \frac{113.14\delta_3}{\sqrt{n-1}} + \frac{C}{n} \end{aligned}$$

for any positive real numbers z, λ, and a constant C. Furthermore, if $\delta_3 \sim \frac{1}{\sqrt{n}}$, then for $n \geq 32$,

$$P(z \leq W_n \leq z + \lambda) \leq \frac{269.0284\lambda}{(1+z)^3} + \frac{608.347}{(1+z)^2\sqrt{n}} + O\left(\frac{1}{n^{\frac{19}{24}}}\right).$$

In this article, auxiliary results are given in Sec. 2 and the main theorem is provided in Sec. 3.

2. Auxiliary Results

Lemma 2.1. *Assume that* $(1+z)^2\delta_3 < \frac{1}{4}$, $\delta_3 \leq \frac{1}{11}$ *and* $n \geq 32$, *then*

$$E\left[\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,k)\right]^4 \leq 0.2333n^4,$$

where $\widehat{Y}_z(i,j) = Y(i,j)\mathbb{I}(|Y(i,j)| \leq 1+z)$.

Proof. Observe that

$$E\left[\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,k)\right]^4 = A_1 + A_2 + A_3 + A_4 + A_5$$

where

$$\begin{aligned}
A_1 &= \sum_{i=1}^{n}\sum_{k=1}^{n} E\widehat{Y}_z^4(i,k)\\
A_2 &= \sum_{i,k}\sum_{\substack{l,m\\(l,m)\neq(i,k)}} E\widehat{Y}_z^3(i,k)\widehat{Y}_z(l,m)\\
A_3 &= \sum_{i,k}\sum_{\substack{l,m\\(l,m)\neq(i,k)}} E\widehat{Y}_z^2(i,k)\widehat{Y}_z^2(l,m)\\
A_4 &= \sum_{i,k}\sum_{\substack{l,m\\(l,m)\neq(i,k)}}\sum_{\substack{p,n\\(p,n)\neq(i,k)\\(p,n)\neq(l,m)}} E\widehat{Y}_z^2(i,k)\widehat{Y}_z(l,m)\widehat{Y}_z(p,n)\\
A_5 &= \sum_{i,k}\sum_{\substack{l,m\\(l,m)\neq(i,k)}}\sum_{\substack{p,n\\(p,n)\neq(i,k)\\(p,n)\neq(l,m)}}\sum_{\substack{r,s\\(r,s)\neq(i,k)\\(r,s)\neq(l,m)\\(r,s)\neq(p,n)}} E\widehat{Y}_z(i,k)\widehat{Y}_z(l,m)\widehat{Y}_z(p,n)\widehat{Y}_z(r,s).
\end{aligned}$$

Moreover,

$$|A_1| \leq (1+z)\sum_{i=1}^{n}\sum_{k=1}^{n}E|Y(i,k)|^3 = (1+z)n\delta_3. \tag{2.1}$$

Note that for $a_i > 0$ and $\alpha_i > 0$, $i \in \{1,2,\ldots,n\}$ with $\alpha_1+\alpha_2+\cdots+\alpha_n = 1$,

$$a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_n^{\alpha_n} \leq \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n, \tag{2.2}$$

and for any non negative integer m and positve integers n and r,

$$\begin{aligned}
E|Y^m(i_1,i_2)Y_z^n(i_1,i_2)| &\leq E|Y^m(i_1,i_2)Y_z^n(i_1,i_2)|\frac{|Y_z(i_1,i_2)|^r}{(1+z)^r}\\
&\leq \frac{E|Y(i_1,i_2)|^{m+n+r}}{(1+z)^r}.
\end{aligned} \tag{2.3}$$

From (1.1),(2.2), (2.3), and the fact that $\widehat{Y}_z = Y - Y_z$ where $Y_z(i,j) = Y(i,j)$ $\mathbb{I}(|Y(i,j)| > 1+z)$, we can show that

$$|A_2 + A_3 + A_4 + A_5| \leq 2(1+z)n\delta_3 + 7n^2\delta_3^2 + 2n^2 + 10n^4\delta_3^2 + 19(1+z)n^3\delta_3 + 3n^3\delta_3^3 + n^4\delta_3^4.$$

Hence, if $(1+z)^2\delta_3 < \frac{1}{4}$, $\delta_3 \leq \frac{1}{11}$ and $n \geq 32$, then this lemma is proved. □

In the rest of this article, we use the following system given by Ho and Chen (1978) and Neammanee and Suntornchost (2005). Let I, K, L, M be uniformly distributed random variables on $\{1, 2, \ldots, n\}$ such that:

$\{I, K, L, M\}$ is independent of $Y(i,j)'s$,

$[(I,K),(L,M)]$ are uniformly distributed on

$\{[(i,k),(l,m)]\,|, i,k,l,m = 1,2,\ldots,n$ and $i \neq k, l \neq m$ and $(i,k) \neq (l,m)\}$,

$(I,K), (L,M)$ and π are mutually independent.

Now, we define some notations:

$$\widetilde{W}_z = W_z - \widehat{S}_{1,z} - \widehat{S}_{2,z} + \widehat{S}_{3,z} + \widehat{S}_{4,z}$$

where

$$\widehat{S}_{1,z} = \widehat{Y}_z(I,\pi(I)), \quad \widehat{S}_{2,z} = \widehat{Y}_z(K,\pi(K)),$$
$$\widehat{S}_{3,z} = \widehat{Y}_z(I,\pi(K)), \quad \widehat{S}_{4,z} = \widehat{Y}_z(K,\pi(I)).$$

It is well-known that $\widetilde{W}_z$ and W_z is an exchangeable pair, i.e., for every $a, b \in \mathbb{R}$

$$P(W \leq a, \widetilde{W} \leq b) = P(W \leq b, \widetilde{W} \leq a)$$

and $\widehat{S}_{i,z}$ for $i = 1,2,3,4$ are identically distributed.

Lemma 2.2. *Let* $\gamma = \max(\frac{n}{4}E|\widetilde{W}_z - W_z|^3, \delta_3)$ *and*

$$U_\gamma = \sum_{i=1}^{n}\sum_{\substack{k\\k\neq i}} |\widehat{Y}_z(i,\pi(i)) + \widehat{Y}_z(k,\pi(k)) - \widehat{Y}_z(i,\pi(k)) - \widehat{Y}_z(k,\pi(i))|$$
$$\times \min\left(\gamma, \sum_{i=1}^{n}\sum_{\substack{k\\k\neq i}} |\widehat{Y}_z(i,\pi(i)) + \widehat{Y}_z(k,\pi(k)) - \widehat{Y}_z(i,\pi(k)) - \widehat{Y}_z(k,\pi(i))|\right).$$

Then:

1. *If* $\delta_3 \leq \frac{1}{11}$ *and* $n \geq 32$, $EU_\gamma \geq 2.2728(n-1) - 5.4778$;
2. *If* $(1+z)^2\delta_3 < \frac{1}{4}$ *and* $n \geq 32$, $VarU_\gamma \leq \frac{23.673n^2\delta_3}{(1+z)^2}$.

Proof. 1. We observe that

$$\begin{aligned}
E(\widetilde{W}_z - W_z)^2 &= \sum_{k=1}^{4} ES_k^2 + 2E\{S_1S_2 - S_1S_3 - S_1S_4 - S_2S_3 - S_2S_4 + S_3S_4\} \\
&\quad - 2E(S_1 + S_2 - S_3 - S_4)(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}) \\
&\quad + E(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z})^2
\end{aligned}$$

and

$$\begin{aligned}
ES_1^2 &= \frac{1}{n}\sum_{i=1}^{n} EY^2(i, \pi(i)) \\
&= \frac{1}{n}\left[EW^2 - \sum_{i=1}^{n}\sum_{\substack{j \\ j\neq i}} EY(i, \pi(i))Y(j, \pi(j))\right] \\
&= \frac{1}{n} - \frac{1}{n^2(n-1)}\sum_{i=1}^{n}\sum_{\substack{j \\ j\neq i}}\sum_{l=1}^{n}\sum_{\substack{m \\ m\neq l}} EY(i, l)EY(j, m) \\
&= \frac{1}{n} - \frac{1}{n^2(n-1)}\sum_{i=1}^{n}\sum_{l=1}^{n}(EY(i, l))^2.
\end{aligned}$$

Let $\delta_2 = \frac{1}{n\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{n} E|Y(i, j)|^2$. Since S_1, S_2, S_3, S_4 are identically distributed,

$$\begin{aligned}
\sum_{k=1}^{4} ES_k^2 &= \frac{4}{n} - \frac{4}{n^2(n-1)}\sum_{i=1}^{n}\sum_{l=1}^{n}(EY(i, l))^2 \\
&\geq \frac{4}{n} - \frac{4\delta_2}{\sqrt{n}(n-1)} \\
&\geq \frac{4}{n} - \frac{4}{n(n-1)},
\end{aligned}$$

where we used $\delta_2 \leq \frac{1}{\sqrt{n}}$ (Neammanee and Rerkruthairat, 2012) in the last inequality. From (1.1), we can show that

$$\begin{aligned}
&2E\{S_1S_2 - S_1S_3 - S_1S_4 - S_2S_3 - S_2S_4 + S_3S_4\} \\
&\quad - 2E(S_1 + S_2 - S_3 - S_4)(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}) \\
&= 2\left\{\frac{2}{n^2(n-1)^2}\sum_{i=1}^{n}\sum_{l=1}^{n}(EY(i, l))^2 + \frac{4}{n^2(n-1)}\sum_{i=1}^{n}\sum_{l=1}^{n}(EY(i, l))^2\right\} \\
&\quad - 2\left\{\frac{4}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} EY(i, j)Y_z(i, j) + \frac{4}{n^2(n-1)^2}\sum_{k=1}^{n}\sum_{m=1}^{n} EY(k, m)EY_z(k, m)\right. \\
&\qquad \left. + \frac{8}{n^2(n-1)}\sum_{i=1}^{n}\sum_{m=1}^{n} EY(i, m)EY_z(i, m)\right\}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
E(\widetilde{W}_z - W_z)^2 \geq{}& \frac{4}{n} - \frac{4}{n(n-1)} \\
&+ 2\left\{\frac{2}{n^2(n-1)^2}\sum_{i=1}^{n}\sum_{l=1}^{n}(EY(i,l))^2 + \frac{4}{n^2(n-1)}\sum_{i=1}^{n}\sum_{l=1}^{n}(EY(i,l))^2\right\} \\
&- 2\left\{\frac{4}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}EY(i,j)Y_z(i,j) + \frac{4}{n^2(n-1)^2}\sum_{k=1}^{n}\sum_{m=1}^{n}EY(k,m)EY_z(k,m)\right. \\
&\left.\quad + \frac{8}{n^2(n-1)}\sum_{i=1}^{n}\sum_{m=1}^{n}EY(i,m)EY_z(i,m)\right\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{4}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}EY(i,j)Y_z(i,j) + \frac{4}{n^2(n-1)^2}\sum_{k=1}^{n}\sum_{m=1}^{n}EY(k,m)EY_z(k,m) \\
&\quad + \frac{8}{n^2(n-1)}\sum_{i=1}^{n}\sum_{m=1}^{n}EY(i,m)EY_z(i,m) \\
&\leq \frac{4}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{E|Y(i,j)|^3}{(1+z)} + \frac{4}{n^2(n-1)^2}\sum_{k=1}^{n}\sum_{m=1}^{n}\frac{E|Y(k,m)|^3}{(1+z)} \\
&\quad + \frac{8}{n^2(n-1)}\sum_{i=1}^{n}\sum_{m=1}^{n}\frac{E|Y(i,m)|^3}{(1+z)} \\
&\leq \frac{4\delta_3}{n} + \frac{4\delta_3}{n(n-1)^2} + \frac{8\delta_3}{n(n-1)} \\
&\leq \frac{0.3636}{n} + \frac{0.7389}{n(n-1)},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
E(\widetilde{W}_z - W_z)^2 &\geq \frac{4}{n} - \frac{4}{n(n-1)} - 2\left\{\frac{0.3636}{n} + \frac{0.7389}{n(n-1)}\right\} \\
&= \frac{3.2728}{n} - \frac{5.4778}{n(n-1)}.
\end{aligned}
$$

From the fact that $\min(a,b) \geq b - \frac{b^2}{4a}$ for any $a, b > 0$, and note that for $n \geq 32$, we can show that

$$
\begin{aligned}
EU_\gamma &= n(n-1)E|\widetilde{W}_z - W_z|\min(\gamma, |\widetilde{W}_z - W_z|) \\
&\geq n(n-1)\left\{E|\widetilde{W}_z - W_z|^2 - \frac{1}{4\gamma}E|\widetilde{W}_z - W_z|^3\right\} \\
&= n(n-1)E|\widetilde{W}_z - W_z|^2 - \frac{n(n-1)}{4\gamma}E|\widetilde{W}_z - W_z|^3
\end{aligned}
$$

$$\geq n(n-1)\left[\frac{3.2728}{n} - \frac{5.4778}{n(n-1)}\right] - (n-1)$$
$$= 2.2728(n-1) - 5.4778.$$

2. By using the idea from Lemma 2.5 in Laipaporn and Neammanee (2006), we can show that

$$VarU_{\gamma} \leq \frac{23.673n^2\delta_3}{(1+z)^2}.$$

□

3. Proof of the Main Result

Let $W_z = \sum_{i=1}^{n} \widehat{Y}_z(i, \pi(i))$. Note that

$$P(z \leq W_n \leq z+\lambda) \leq P(W_n \neq W_z) + P(z \leq W_z \leq z+\lambda) \tag{3.1}$$

and from inequality (3.2) in Neammanee and Rattanawong (2009), we have

$$P(W_n \neq W_z) \leq \frac{\delta_3}{(1+z)^3}. \tag{3.2}$$

Hence, to prove Theorem 1.2, it suffices to bound $P(z \leq W_z \leq z+\lambda)$.

Assume that $n \geq 32$. If $(1+z)^2\delta_3 \geq \frac{1}{4}$, by using the same argument as Lemma 2.1(3) in Neammanee and Rerkruthairat (2012), we obtain that

$$EW_z^4 \leq 1.7963 + 12.6616(1+z)\delta_3 + 0.0125n\delta_3^2. \tag{3.3}$$

Thus,

$$\begin{aligned}
P(z \leq W_z \leq z+\lambda) &\leq P(z \leq W_z) \\
&= P(1+z \leq W_z + 1) \\
&\leq \frac{E|W_z+1|^4}{(1+z)^4} \\
&\leq \frac{8EW_z^4}{(1+z)^4} + \frac{8}{(1+z)^4} \\
&\leq \frac{22.3704}{(1+z)^4} + \frac{101.2928(1+z)\delta_3}{(1+z)^4} + \frac{0.1n\delta_3^2}{(1+z)^4} \\
&\leq \frac{190.7744\delta_3}{(1+z)^2} + \frac{0.4n\delta_3^3}{(1+z)^2}. \qquad (3.4)
\end{aligned}$$

Suppose that $(1+z)^2\delta_3 < \frac{1}{4}$ and $\delta_3 \leq \frac{1}{11}$. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} 0 & \text{if } t < z-\gamma, \\ (1+t+\gamma)^3(t-z+\gamma) & \text{if } z-\gamma \leq t \leq z+\lambda+\gamma, \\ (1+t+\gamma)^3(\lambda+2\gamma) & \text{if } t > z+\lambda+\gamma. \end{cases}$$

By the same argument as (1) in Laipaporn and Sungkamongkol (2009), we can show that

$$P(z \le W_z \le z+\lambda) \le \frac{4}{(1+z)^3} EW_z g(W_z) - \frac{4}{(1+z)^3}\Delta g(W_z) + P(U_\gamma \le n-1). \tag{3.5}$$

Note that, from Lemma 2.2,

$$EU_\gamma \ge 2.2728(n-1) - 5.4778$$

and

$$VarU_\gamma \le \frac{23.6735 n^2 \delta_3}{(1+z)^2}.$$

From the above facts, we can conclude that

$$\begin{aligned} P(U_\gamma \le n-1) &= P(EU_\gamma - U_\gamma \ge 1.2728(n-1) - 5.4778) \\ &\le \frac{VarU_\gamma}{(1.2728(n-1)-5.4778)^2} \\ &\le \frac{VarU_\gamma}{1.2012(n-1)^2} \\ &\le \frac{21.0002\delta_3}{(1+z)^2}. \end{aligned} \tag{3.6}$$

Using the same argument as Lemma 2.1(2) in Neammanee and Rerkruthairat (2012) we have $EW_z^2 < 1.0645$, and from (3.3) in case of $(1+z)^2\delta_3 < \frac{1}{4}$, we also obtain that $EW_z^4 \le 5.8177 + 2.8441 n\delta_3^2$. Thus, from the definition of g

$$\begin{aligned} EW_z g(W_z) &\le (\lambda + 2\delta_3) E|W_z||(1+\delta_3) + W_z|^3 \\ &\le 4(\lambda+2\delta_3)E|W_z|((1+\delta_3)^3 + |W_z|^3) \\ &\le 5.193(\lambda+2\delta_3)\{EW_z^2\}^{1/2} + 4(\lambda+2\delta_3)EW_z^4 \\ &\le (28.6286 + 11.3764 n\delta_3^2)(\lambda + 2\delta_3). \end{aligned} \tag{3.7}$$

By the same argument as Lemma 2.1(1) in Neammanee and Rerkruthairat (2012),

$$E\left[\sum_{i=1}^n \sum_{k=1}^n \widehat{Y}_z(i,k)\right]^2 \le n + n^2\delta_3^2.$$

As a result of the above fact and Lemma 2.1,

$$\begin{aligned} |\Delta g(W_z)| &= \left| \frac{1}{n} E g(W_z) \sum_{i=1}^n \sum_{k=1}^n \widehat{Y}_z(i,\pi(k)) \right| \\ &\le \frac{1}{n}(\lambda + 2\delta_3) E \left| (1+\delta_3+W_z)^3 \sum_{i=1}^n \sum_{k=1}^n \widehat{Y}_z(i,\pi(k)) \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{4}{n}(\lambda+2\delta_3)E\left|(1+\delta_3)^3\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,\pi(k))\right| \\
&\quad + 4(\lambda+2\delta_3)E|W_z|^3\left|\frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,\pi(k))\right| \\
&\leq 4(\lambda+2\delta_3)(1+\delta_3)^3\frac{1}{n}\left\{E\left[\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,k)\right]^2\right\}^{\frac{1}{2}} \\
&\quad + 4(\lambda+2\delta_3)\{EW_z^4\}^{\frac{3}{4}}\left\{E\left|\frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,k)\right|^4\right\}^{\frac{1}{4}} \\
&\leq 1.0334(\lambda+2\delta_3)+3(\lambda+2\delta_3)\{EW_z^4\} \\
&\quad + (\lambda+2\delta_3)\left\{E\left|\frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,k)\right|^4\right\} \\
&\leq (18.7198+8.5323n\delta_3^2)(\lambda+2\delta_3). \qquad (3.8)
\end{aligned}
$$

Thus, from (3.5)–(3.8),

$$
\begin{aligned}
P(z \leq W_z \leq z+\lambda) &\leq \frac{(189.3936+79.6348n\delta_3^2)\lambda}{(1+z)^3} \\
&\quad + \frac{(399.7874+159.2696n\delta_3^2)\delta_3}{(1+z)^2}. \qquad (3.9)
\end{aligned}
$$

Suppose that $(1+z)^2\delta_3 < \frac{1}{4}$ and $\delta_3 > \frac{1}{11}$. Hence, $\frac{1}{(1+z)^2} > \frac{4}{11}$. From this fact and the result of Neammanee and Rerkruthairat (2012) as follows:

$$
\sup_{z\in\mathbb{R}}|P(W_n \leq z)-\Phi(z)| \leq 69.58\delta_3+\frac{8.78}{\sqrt{n}}+\frac{C_1\delta_3^{\frac{3}{4}}(1+n\delta_3^2)^{\frac{1}{4}}}{n^{\frac{5}{12}}}+\frac{56.57\delta_3}{\sqrt{n-1}}+\frac{C_1}{n}
$$

where C_1 is a constant and Φ is the standard normal distribution, we have

$$
\begin{aligned}
P(z \leq W_n \leq z+\lambda) &\leq |P(W_n \leq z+\lambda)-\Phi(z+\lambda)|+|P(W_n < z)-\Phi(z)| \\
&\quad + |\Phi(z+\lambda)-\Phi(z)| \\
&\leq 139.16\delta_3+\frac{17.56}{\sqrt{n}}+\frac{C_2\delta_3^{\frac{3}{4}}(1+n\delta_3^2)^{\frac{1}{4}}}{n^{\frac{5}{12}}}+\frac{113.14\delta_3}{\sqrt{n-1}}+\frac{C_2}{n}+\frac{\lambda}{e^{z^2/2}} \\
&\leq \frac{382.69\delta_3}{(1+z)^2}+\frac{48.29}{\sqrt{n}(1+z)^2}+\frac{C_2\delta_3^{\frac{3}{4}}(1+n\delta_3^2)^{\frac{1}{4}}}{n^{\frac{5}{12}}} \\
&\quad + \frac{113.14\delta_3}{\sqrt{n-1}}+\frac{C_2}{n}+\frac{C_3\lambda}{(1+z)^3} \qquad (3.10)
\end{aligned}
$$

where C_2 is a constant and

$$C_3 = \begin{cases} 13; & 0 \leq x \leq 1.3 \\ 5.1813; & x > 1.3. \end{cases}$$

Hence, from (3.1), (3.2), (3.4), (3.9), and (3.10), the main theorem is obtained.

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