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THE FLOOR OF THE ARITHMETIC MEAN OF THE CUBE ROOTS OF THE FIRST *n* INTEGERS

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Abstract

Zacharias ['Proof of a conjecture of Merca on an average of square roots', *College Math. J.* **49** (2018), 342–345] proved Merca's conjecture that the arithmetic means $(1/n)\sum_{k=1}^{n}\sqrt{k}$ of the square roots of the first n integers have the same floor values as a simple approximating sequence. We prove a similar result for the arithmetic means $(1/n)\sum_{k=1}^{n}\sqrt[3]{k}$ of the cube roots of the first n integers.

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1. Introduction

Sums of powers of positive integers have fascinated mathematicians for a long time. In 1631, Faulhaber gave the formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_j n^{p-j+1}$$

for any positive integer p, where the B_j are Bernoulli numbers with $B_1 = -\frac{1}{2}$. Gould [1] and Merca [2] formulated this sum in term of Stirling numbers,

$$\sum_{k=1}^{n} k^p = \sum_{j=0}^{p}(-1)^{j-1} j \left\{ {n+p-j \atop n} \right\} \left[{n+1 \atop n-j+1} \right].$$

Ramanujan [3] gave a formula for the sum of square roots of the first n integers,

$$\sum_{k=1}^{n}\sqrt{k} = \frac{2}{3}n\sqrt{n} + \frac{1}{2}\sqrt{n} + \frac{1}{6}\sum_{j=0}^{\infty}\frac{1}{(\sqrt{n+j}+\sqrt{n+j+1})^3} - C, \quad C = \frac{1}{4\pi}\sum_{j=1}^{\infty}\frac{1}{j\sqrt{j}}.$$

Shekatkar [4] extended this result to rth roots,

$$\sum_{k=1}^{n}\sqrt[r]{k} = \frac{r}{r+1}(n+1)\sqrt[r]{n+1} - \frac{1}{2}\sqrt[r]{n+1} - \phi_n(r),$$

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where $\phi_n(r)$ depends on the parameters n and r and satisfies $0 \le \phi_n(r) < \frac{1}{2}$. Recently, Wihler [5] gave another expression for this sum,

$$\sum_{k=v}^{n} \sqrt[r]{k} = \frac{r}{r+1}\sqrt[r]{n+1}\left(n + \frac{1-1/r}{2}\right) - \frac{r}{r+1}\sqrt[r]{v}\left(v - \frac{1+1/r}{2}\right) - \frac{\delta_{v,n,r}}{12r} \tag{1.1}$$

with $\delta_{v,n,1} = 0$ and $\sigma_r(v+2, n+2) < \delta_{v,n,r} < \sigma_r(v,n)$ for $r > 1$, where

$$\sigma_r(v,n) = \begin{cases} 2 - \dfrac{1}{r} - n^{-1+1/r} & \text{if } v = 1, \\ (v-1)^{-1+1/r} - n^{-1+1/r} & \text{if } v \ge 2. \end{cases}$$

In [2], Merca established another approximation for the arithmetic mean of the square roots of the first n integers and conjectured that the floor values of the average and the approximation are the same, that is,

$$\left\lfloor \frac{1}{n}\sum_{k=1}^{n} \sqrt{k} \right\rfloor = \left\lfloor \left(\frac{2}{3} + \frac{1}{6n}\right)\sqrt{n+1} \right\rfloor.$$

Zacharias [6] proved this conjecture by constructing a step function which has steps at two types of abscissas depending on numbers modulo 2. In this paper, we present a similar theorem for cube roots.

THEOREM 1.1. *For any positive integer n,*

$$\left\lfloor \frac{1}{n}\sum_{k=1}^{n} \sqrt[3]{k} \right\rfloor = \left\lfloor \left(\frac{3}{4} + \frac{1}{4n}\right)\sqrt[3]{n+1} - \frac{1}{4n} \right\rfloor. \tag{1.2}$$

The idea of our proof is to divide n into nine cases based on numbers modulo 9 and to prove that, in each case, the values on the left- and right-hand sides of Equation (1.2) have the same floor.

2. Proof of Theorem 1.1

For $x \ge 1$, define

$$A(x) = \left(\frac{3}{4} + \frac{1}{4x}\right)\sqrt[3]{x+1} - \frac{1}{4x} \quad \text{and} \quad L(x) = \left(\frac{3}{4} + \frac{1}{4x}\right)\sqrt[3]{x+1} - \frac{11}{36x}.$$

From their derivatives, $A(x)$ and $L(x)$ are increasing functions for $x \ge 1$.

From (1.1),

$$\frac{1}{n}\sum_{k=1}^{n} \sqrt[3]{k} = \left(\frac{3}{4} + \frac{1}{4n}\right)\sqrt[3]{n+1} - \frac{1}{4n} - \frac{\delta_{1,n,3}}{36n},$$

where $2^{-2/3} - n^{-2/3} < \delta_{1,n,3} < \frac{5}{3} - n^{-2/3}$ and $0 < \delta_{1,n,3} < 2$ for $n > 2$. Consequently,

$$A(n) > \frac{1}{n}\sum_{k=1}^{n} \sqrt[3]{k} > L(n) \quad \text{for } n > 2.$$

In order to prove the main theorem, we need the following lemmas.

Lemma 2.1. *$S_n = (1/n)\sum_{k=1}^{n} \sqrt[3]{k}$ is an increasing sequence.*

Proof. The inequality $S_n \le S_{n+1}$ is equivalent to $\sum_{k=1}^{n} \sqrt[3]{k} \le n\sqrt[3]{n+1}$. We prove the latter assertion using mathematical induction. The assertion is clear for $n = 1$. To show the inductive step, assume that $\sum_{k=1}^{n} \sqrt[3]{k} \le n\sqrt[3]{n+1}$. Then

$$\sum_{k=1}^{n+1} \sqrt[3]{k} = \sum_{k=1}^{n} \sqrt[3]{k} + \sqrt[3]{n+1} \le (n+1)\sqrt[3]{n+1} < (n+1)\sqrt[3]{n+2}.$$

It follows by induction that $\{S_n\}$ is an increasing sequence. □

For $k \in \mathbb{N} \cup \{0\}$ and $j = 1, 2$, we define the sets $B_{j,k}$ as follows:

$$B_{1,k} = \begin{cases} \left[\dfrac{64(9k+1)^3 - 37}{27}, \dfrac{64(9k+2)^3 - 53}{27}\right] \cap \mathbb{N} & \text{if } k = 0, 1, 2, \\ \left[\dfrac{64(9k+1)^3 - 37}{27}, \dfrac{64(9k+2)^3 - 80}{27}\right] \cap \mathbb{N} & \text{if } k \ge 3, \end{cases}$$

$$B_{2,k} = \begin{cases} \left[\dfrac{64(9k+2)^3 - 26}{27}, \dfrac{64(9k+3)^3 - 54}{27}\right] \cap \mathbb{N} & \text{if } k = 0, 1, 2, \\ \left[\dfrac{64(9k+2)^3 - 53}{27}, \dfrac{64(9k+3)^3 - 54}{27}\right] \cap \mathbb{N} & \text{if } k \ge 3 \end{cases}$$

and, for $j = 3, 4, \dots, 9$, we define the sets $B_{j,k}$ by

$$B_{j,k} = \left[\frac{64(9k+j)^3 - b_j}{27}, \frac{64(9k+j+1)^3 - b_{j+1} - 27}{27}\right],$$

where

$$b_j = \begin{cases} 27 & \text{if } j \equiv 0 \pmod 3, \\ 2(25 - i) & \text{if } j \not\equiv 0 \pmod 3 \text{ and } j = 2^i, \\ j(5 + (-1)^i i) & \text{if } j \text{ is prime and } j \equiv i \pmod 3,\ 0 < i < 3 \end{cases}$$

and $b_{10} = 37$.

The following lemma shows that the classes $B_{j,k}$ partition $\mathbb{N}$. This will allow us to divide the proof of Theorem 1.1 into nine cases.

Lemma 2.2. *The set $\{B_{j,k} \mid 1 \le j \le 9, k \ge 0\}$ forms a partition of $\mathbb{N}$, that is,*

$$\mathbb{N} = \bigcup_{k=0}^{\infty} \bigcup_{j=1}^{9} B_{j,k}, \quad \textit{where } B_{j,k} \cap B_{j+1,k} = \emptyset,\ B_{9,k} \cap B_{1,k+1} = \emptyset.$$

Proof. Notice that

$$\min B_{1,0} = \frac{64(9(0)+1)^3 - 37}{27} = 1$$

and it is easy to check that all the boundary points of the $B_{j,k}$ are integers. It is obvious from the definition of $B_{j,k}$ that $\max B_{j,k} + 1 = \min B_{j+1,k}$ for $1 \le j \le 8$ and $\max B_{9,k} + 1 = \min B_{1,k+1}$. Hence, $\{B_{j,k}\}$ partitions $\mathbb{N}$. □

For any $n \in B_{j,k}$, to prove that S_n and $A(n)$ are in the same interval $[9k + j, 9k + j + 1)$, we use the facts that $L(n) < S_n < A(n)$ and that both $L(n)$ and $A(n)$ are increasing sequences. Consequently, for $n_1 = \min B_{j,k}$ and $n_2 = \max B_{j,k}$, it is sufficient to show that $L(n_1) > 9k + j$ and $A(n_2) < 9k + j + 1$ by considering the sign of the coefficients of certain Taylor expansions. In the case of $n \in B_{1,k}$, however, this technique is not applicable for some numbers k. For these basic numbers k, we use a simple direct calculation instead.

Proof of Theorem 1.1. Let $n \in \mathbb{N}$.

Case 1. $n \in B_{1,k}$.

Case 1.1. $k = 0$. Observe that

$$1 = \frac{64(1)^3 - 3}{27} \leq n \leq \frac{64(2)^3 - 53}{27} = 17.$$

Since S_n and $A(n)$ are increasing sequences,

$$1 = S_1 \leq S_n \leq S_{17} \approx 1.9880 \quad \text{and} \quad 1.0099 \approx A(1) \leq A(n) \leq A(17) \approx 1.9894.$$

Hence, $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 1$ for $1 \leq n \leq 17$.

Case 1.2. $k \geq 1$. Let

$$n_1 = \frac{64(9k + 1)^3 - 37}{27}.$$

We will show that $L(n_1) > 9k + 1$. Substitute n_1 into the following expression and expand it as a Taylor series about 1:

$$\begin{aligned}
&(36n_1(9k + 1) + 11)^3 - (n_1 + 1)(27n_1 + 9)^3 \\
&\quad = -140912433282709 - 1163293726126932(k - 1) - 4266503508623472(k - 1)^2 \\
&\qquad - 9124406608540608(k - 1)^3 - 12539857656248064(k - 1)^4 \\
&\qquad - 11485195678476288(k - 1)^5 - 7010519795527680(k - 1)^6 \\
&\qquad - 2750059732008960(k - 1)^7 - 629107509362688(k - 1)^8 \\
&\qquad - 63945157902336(k - 1)^9 < 0.
\end{aligned}$$

This gives $36n_1(9k + 1) + 11 < \sqrt[3]{n_1 + 1}(27n_1 + 9)$ and, by a simple calculation,

$$9k + 1 < \sqrt[3]{n_1 + 1}\left(\frac{3}{4} + \frac{1}{4n_1}\right) - \frac{11}{36n_1} = L(n_1).$$

Let

$$n_2 = \begin{cases} \dfrac{64(9k + 2)^3 - 53}{27} & \text{if } k = 1, 2, \\[2ex] \dfrac{64(9k + 2)^3 - 80}{27} & \text{if } k \geq 3. \end{cases}$$

We will prove that $A(n_2) < 9k + 2$. For $k = 1, 2$,

$$(4n_2(9k + 2) + 1)^3 - (n_2 + 1)(3n_2 + 1)^3 = \begin{cases} 26036934837 & \text{if } k = 1, \\ 72986823793 & \text{if } k = 2. \end{cases}$$

Hence, $(4n_2(9k+2)+1)^3 - (n_2+1)(3n_2+1)^3 > 0$. For $k = 3, 4, 5, \ldots,$

$$\begin{aligned}(4n_2(9k+2)+1)^3 &- (n_2+1)(3n_2+1)^3\\ &= 134154289152k^9 + 279918084096k^8 + 259135635456k^7 + 138919919616k^6\\ &\quad + 47252865024k^5 + 10507311360k^4 + 1516397696k^3\\ &\quad + 135961344k^2 + 6837440k + 146656 > 0.\end{aligned}$$

This implies that

$$9k+2 > \sqrt[3]{n_2+1}\left(\frac{3}{4} + \frac{1}{4n_2}\right) - \frac{1}{4n_2} = A(n_2).$$

Since $A(n)$ and $L(n)$ are increasing, for $n_1 \le n \le n_2$,

$$9k+1 < L(n) \le S_n \le A(n) < 9k+2.$$

Hence, $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 9k+1$ for $k \ge 1$.

Case 2. $n \in B_{2,k}$. Let

$$n_1 = \begin{cases} \dfrac{64(9k+2)^3 - 26}{27} & \text{for } k = 0, 1, 2,\\[2ex] \dfrac{64(9k+2)^3 - 53}{27} & \text{for } k \ge 3.\end{cases}$$

For $k = 0, 1, 2,$

$$\begin{aligned}(36n_1(9k+2)+11)^3 &- (n_1+1)(27n_1+9)^3\\ &= -105321436545024k^9 - 200298803429376k^8 - 168848652238848k^7\\ &\quad - 82685802627072k^6 - 25886537404416k^5 - 5366022080256k^4\\ &\quad - 735591839616k^3 - 64231766208k^2 - 3238636392k\\ &\quad - 71778682 < 0.\end{aligned}$$

For $k \ge 3$, using the Taylor series expansion about 3,

$$\begin{aligned}(36n_1(9k+2)+11)^3 &- (n_1+1)(27n_1+9)^3\\ &= -20723370376131937 - 95110224549962868(k-3)\\ &\quad - 164307456860697648(k-3)^2 - 152480690365127616(k-3)^3\\ &\quad - 86582810797661952(k-3)^4 - 31713445228234752(k-3)^5\\ &\quad - 7563596651458560(k-3)^6 - 1139310460403712(k-3)^7\\ &\quad - 98738846760960(k-3)^8 - 3761479876608(k-3)^9 < 0.\end{aligned}$$

As in Case 1, this gives $9k+2 < L(n_1)$. Let

$$n_2 = \frac{64(9k+3)^3 - 54}{27}.$$

We claim that $A(n_2) < 9k+3$. Substituting n_2 into the following expression:

$$\begin{aligned}
&(4n_2(9k+3)+1)^3 - (n_2+1)(3n_2+1)^3 \\
&\quad = -80244904034304k^{11} - 285315214344192k^{10} - 460995415769088k^9 \\
&\qquad - 446505462595584k^8 - 287877045288960k^7 - 129650051727360k^6 \\
&\qquad - 41596420313088k^5 - 9502311626496k^4 - 1513938218112k^3 \\
&\qquad - 160144814016k^2 - 10118551896k - 289206316 < 0
\end{aligned}$$

as before, this gives $A(n_2) < 9k+3$. Thus, for $n_1 \le n \le n_2$,

$$9k+2 < L(n) \le S_n \le A(n) < 9k+3.$$

Hence, $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 9k+2$.

Case 3. $n \in B_{3,k}$. Let

$$n_1 = \frac{64(9k+3)^3 - 27}{27}.$$

By direct calculation,

$$\begin{aligned}
&(36n_1(9k+3)+11)^3 - (n_1+1)(27n_1+9)^3 \\
&\quad = -101559956668416k^9 - 294335800344576k^8 - 378655640911872k^7 \\
&\qquad - 283684804374528k^6 - 136343850006528k^5 - 43577890742016k^4 \\
&\qquad - 9259059419712k^3 - 1260645061680k^2 - 99771741900k \\
&\qquad - 3496110625 < 0.
\end{aligned}$$

Consequently, $9k+3 < L(n_1)$. Let

$$n_2 = \frac{64(9k+4)^3 - 73}{27}.$$

As before, we obtain $A(n_2) < 9k+4$ and conclude that $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 9k+3$.

For the remaining cases, we use arguments similar to those in Case 3 to show that

$$\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 9k + j \quad \text{for } j = 4, \dots, 9. \qquad \square$$

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