Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 25, 2019, No. 2, 36–39
DOI: 10.7546/nntdm.2019.25.2.36-39

Remark on s_k, t-Jacobsthal numbers

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Received: 6 January 2019 **Revised:** 19 March 2019 **Accepted:** 8 April 2019

Abstract: A new generalization of the Jacobsthal numbers is introduced and the properties of the new numbers are studied.
Keywords: Jacobsthal numbers, s_k, t-Jacobsthal numbers.
2010 Mathematics Subject Classification: 11B37.

1 Introduction

The n^{th} Jacobsthal number $(n \geq 0)$ [1] is defined by

$$J_n = \frac{2^n - (-1)^n}{3}.$$

In 2011, Krassimir T. Atanassov [2] generalized to the form

$$J_n^s = \frac{s^n - (-1)^n}{s+1},$$

where n is non-negative number and s is a positive real number.
And he also gave the form

$$J_n^{s,t} = \frac{s^n - (-t)^n}{s+t},$$

where n is non-negative integer and $s \neq -t$ is arbitrary real number.

In 2014, Alous Cesar F. Bueno [4] introduced s_k-Jacobsthal numbers and studied their properties. The n^{th} s_k-Jacobsthal number is given by

$$J_n^{s_k} = \frac{s_k^n - (-1)^n}{s_{k+1}},$$

where $s_k \neq 0$ and $s_{k+1} \neq 0, 1$ are the k^{th} and $(k+1)^{th}$ terms of the real sequence $\{s_k\}_{k=0}^{\infty}$.

The purpose of this paper is to generate the new generalization of the Jacobsthal number and investigate their properties.

2 Main Results

Definition 2.1. *The n^{th} s_k, t-Jacobsthal number is given by*

$$J_n^{s_k,t} = \frac{s_k^n - (-t)^n}{s_k + t},$$

where n is non-negative integer and $s_k \neq t$ is the k^{th} terms of the real sequence $\{s_k\}_{k=0}^{\infty}$

The first six members of the sequence $\{J_n^{s_k,t}\}$ with respect to n are

n	0	1	2	3	4	5
$J_n^{s_k,t}$	0	1	$s_k - t$	$s_k^2 - s_k t + t^2$	$s_k^3 - s_k^2 t + s_k t^2 - t^3$	$s_k^4 - s_k^3 t + s_k^2 t^2 - s_k t^3 + t^4$

Theorem 2.2. *For non-negative integer a*

$$J_{n+a}^{s_k,t} = s_k^a J_n^{s_k,t} + (-t)^n J_a^{s_k,t}.$$

Proof. Since

$$\begin{aligned} J_{n+a}^{s_k,t} &= \frac{s_k^{n+a} - (-t)^{n+a}}{s_k + t} \\ &= \frac{s_k^a s_k^n - (-t)^n (-t)^a}{s_k + t} \\ &= \frac{s_k^a (s_k^n - (-t)^n)}{s_k + t} + \frac{(-t)^n (s_k^a - (-t)^a}{s_k + t} \\ &= s_k^a J_n^{s_k,t} + (-t)^n J_a^{s_k,t}. \end{aligned}$$

□

Corollary 2.2.1. *If $a = 1$ and $a = 2$ then*

$$J_{n+1}^{s_k,t} = s_k J_n^{s_k,t} + (-t)^n$$

and

$$J_{n+2}^{s_k,t} = s_k^2 J_n^{s_k,t} + (-t)^n (s_k - t).$$

Theorem 2.3.

$$J_{2n}^{s_k,t} = \begin{cases} (s_k^n + t^n) J_n^{s_k,t} & \textit{if } n \textit{ is even} \\ (s_k^n - t^n) J_n^{s_k,t} & \textit{if } n \textit{ is odd .} \end{cases}$$

Proof.

$$\begin{aligned}
J_{2n}^{s_k,t} &= \frac{s_k^{2n} - (-t)^{2n}}{s_k + t} \\
&= \frac{s_k^{2n} - t^{2n}}{s_k + t} \\
&= \frac{(s_k^n - t^n)(s_k^n + t^n)}{s_k + t} \\
&= \begin{cases} (s_k^n + t^n)\frac{(s_k^n - (-t)^n)}{s_k + t} & \text{if } n \text{ is even} \\ (s_k^n - t^n)\frac{(s_k^n - (-t)^n)}{s_k + t} & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} (s_k^n + t^n)J_n^{s_k,t} & \text{if } n \text{ is even} \\ (s_k^n - t^n)J_n^{s_k,t} & \text{if } n \text{ is odd .} \end{cases}
\end{aligned}$$

□

Theorem 2.4.

$$(s_k + t)J_n^{s_k,t}J_n^{s_k,t} = 2s_k^n J_n^{s_k,t} - J_{2n}^{s_k,t}.$$

Proof.

$$\begin{aligned}
(s_k + t)J_n^{s_k,t}J_n^{s_k,t} &= \frac{(s_k^n - (-t)^n)(s_k^n - (-t)^n)}{s_k + t} \\
&= \frac{s_k^{2n} + (-t)^{2n} - 2(-t)^n s_k^n}{s_k + t} \\
&= \frac{2s_k^{2n} - s_k^{2n} + (-t)^{2n} - 2(-t)^n s_k^n}{s_k + t} \\
&= \frac{2s_k^{2n} - 2(-t)^n s_k^n}{s_k + t} - \frac{s_k^{2n} - (-t)^{2n}}{s_k + t} \\
&= \frac{2s_k^n(s_k^n - (-t)^n)}{s_k + t} - \frac{s_k^{2n} - (-t)^{2n}}{s_k + t} \\
&= 2s_k^n J_n^{s_k,t} - J_{2n}^{s_k,t}.
\end{aligned}$$

□

Theorem 2.5.

$$J_{n+1}^{s_k,t} + J_n^{s_k,t} = s_k J_n^{s_k,t} + J_n^{s_k,t} + (-t)^n.$$

Proof. Note that $s_k^2 J_n^{s_k,t} + s_k t J_n^{s_k,t} = s_k^{n+1} - s_k(-t)^n$.

$$\begin{aligned}
J_{n+1}^{s_k,t} + J_n^{s_k,t} &= \frac{s_k^{n+1} - (-t)^{n+1}}{s_k + t} + \frac{s_k^n - (-t)^n}{s_k + t} \\
&= \frac{s_k^2 J_n^{s_k,t} + s_k t J_n^{s_k,t} + s_k(-t)^n - (-t)^{n+1} + s_k^n - (-t)^n}{s_k + t} \\
&= \frac{s_k J_n^{s_k,t}(s_k + t) + (-t)^n(s_k + t) + s_k^n - (-t)^n}{s_k + t} \\
&= s_k J_n^{s_k,t} + J_n^{s_k,t} + (-t)^n.
\end{aligned}$$

□

Theorem 2.6.

$$J_{n+1}^{s_k,t} - J_n^{s_k,t} = s_k J_n^{s_k,t} - J_n^{s_k,t} + (-t)^n.$$

Proof.

$$\begin{aligned}
J_{n+1}^{s_k,t} - J_n^{s_k,t} &= \frac{s_k^{n+1} - (-t)^{n+1}}{s_k + t} - \frac{s_k^n - (-t)^n}{s_k + t} \\
&= \frac{s_k^2 J_n^{s_k,t} + s_k t J_n^{s_k,t} + s_k(-t)^n - (-t)^{n+1} - s_k^n + (-t)^n}{s_k + t} \\
&= \frac{s_k J_n^{s_k,t}(s_k + t) + (-t)^n(s_k + t) - (s_k^n - (-t)^n)}{s_k + t} \\
&= s_k J_n^{s_k,t} - J_n^{s_k,t} + (-t)^n. \qquad \square
\end{aligned}$$

Theorem 2.7.

$$J_n^{s_k,t} J_{n+1}^{s_k,t} = \frac{2s_k^{2n+1}}{s_k + t} - J_{2n+1}^{s_k,t} + \frac{(-ts_k)^n(1 - s_k)}{s_k + t}.$$

Proof.

$$\begin{aligned}
J_n^{s_k,t} J_{n+1}^{s_k,t} &= \left[\frac{s_k^n - (-t)^n}{s_k + t}\right]\left[\frac{s_k^{n+1} - (-t)^{n+1}}{s_k + t}\right] \\
&= \frac{(s_k^n - (-t)^n)(s_k^{n+1} - (-t)^{n+1})}{(s_k + t)^2} \\
&= \frac{s_k^{2n+1} - (-t)^n s_k^{n+1} + (-t)^n s_k^n + (-t)^{2n+1}}{(s_k + t)^2} \\
&= \left[\frac{s_k^{2n+1} + (-t)^{2n+1}}{s_k + t}\right] + \frac{(-t)^n s_k^n(s_k) + (-t)^n s_k^n}{s_k + t} \\
&= \frac{2s_k^{2n+1} - s_k^{2n+1} + (-t)^{2n+1}}{s_k + t} + \frac{(-ts_k)^n(1 - s_k)}{s_k + t} \\
&= \frac{2s_k^{2n+1}}{s_k + t} - J_{2n+1}^{s_k,t} + \frac{(-ts_k)^n(1 - s_k)}{s_k + t}. \qquad \square
\end{aligned}$$

Acknowledgements

I would like to give a thanks to Jatupat for helping submission process and this work is supported by Faculty of Science, Burapha University, Thailand.

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